VOL. 32, NO. 1, SEPT.-OCT., 1958

# MATHEMATICS

# magazine

# MATHEMATICS MAGAZINE

**Formerly National Mathematics Magazine, founded by S. T. Sanders.**




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The Mathematics Magazine is published at Pacoima, California by the managing editor, bi-monthly except July-August. Ordinary subscriptions are 1 year $3.00; 2 years $5.75; 3 years $8.50; 4 years $11.00; 5 years $13.00. Sponsoring subscriptions are $10.00; single copies 65¢, reprints, bound 1¢ per page plus 10¢ each, provided your order is placed before your article goes to press.

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Entered as second-class matter, March 23, 1948, at the Post Office, Pacoima, California, under act of Congress of March 3, 1876.

MATHEMATICS MAGAZINE

VOL. 32, No. 1, Sept.-Oct., 1958

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# ON THE TRANSPOSE-CONNECTIVITY OF GRAPHS

Paul Kelly and David Merriell

If $A$ is a finite graph in which two points (vertices) have at most one join, then the *transpose* of $A$ is the graph $T(A)$ obtained by reversing the join and non-join relations in $A$. (It is assumed that a point cannot join itself.) A graph is connected if there is a path joining any two distinct points. It is easy to argue that if a graph is disconnected, then its transpose is connected. We give here necessary and sufficient conditions for both $A$ and $T(A)$ to be connected, that is, for the graph to have *transpose-* or *T-connectivity*.

If $p_1, p_2, \cdots, p_n$ denote the vertices of a graph $A$, we use $c(p_1, p_2, \cdots, p_j)$, $j<n$, to designate the subgraph of $A$ with vertices $p_{j+1}, \cdots, p_n$ and all join relations between these vertices which exist in $A$. If $A$ is connected, a vertex $p_i$ is here called an *interior* or an *exterior* point of $A$ according as $c(p_i)$ is disconnected or connected. (Elsewhere, an interior point has been called an articulation or cut point.) The degree of a point is the number of its joins, so an end point has degree 1. The order of a graph is the number of its vertices.

THEOREM 1. *A connected graph of order $n \geqq 2$ has at least two exterior points.*

PROOF. Let $A$ be a connected graph of order $n$. If $n=2$, both points are exterior since an isolated point is a connected graph. Assume the theorem true for $n=k$ and let $A$ have order $k+1$. If all points of $A$ are exterior, the theorem is true. Therefore, assume $A$ has an interior point $p$. If two connected components of $c(p)$ consist of isolated points, these must be end points in $A$ and hence exterior points so the theorem would be true. Hence, assume at least one component has more than one point. By the induction hypothesis, it has at least two exterior points $q$ and $r$. If $p$ joins both $q$ and $r$, each is exterior in $A$ because the removal of either would still leave the component connected to $p$ via the other. If $p$ does not join $q$, then $q$ is exterior in $A$. Similarly, any other component of $c(p)$ must contain at least one exterior point of $A$. Since there are at least two components, $A$ has at least two exterior points.

The next two theorems are readily established.

THEOREM 2. *A necessary and sufficient condition for a graph of order $n \geqq 3$ to be connected is that two of its $(n-1)$-order subgraphs be connected.*

THEOREM 3. *If $p$ is an interior point of $A$, then it is an exterior point*

*of $T(A)$.*

We now show,

THEOREM 4. *A connected graph of order $n \geqq 2$ has exactly two exterior points if and only if it is a simple path.*

PROOF. If $A$ is a simple path, it has exactly two exterior points, its end points. Conversely, if $A$ is connected of order 2, each point is exterior. Assume the theorem true for order $k$. Let $A$ be a graph of order $k+1$ with exactly two exterior points $p$ and $q$. Since $c(p)$is connected, it has at least two exterior points. Call one of these $r$. Then $c(p, r)$ is connected. If $p$ joins any point of $c(p)$ other than $r$, $c(r)$ is connected and $r$ is exterior in $A$. Therefore, the only possibility for an exterior point in $c(p)$ to be an interior point of $A$ is that $p$ be joined to it and to no other point. Since $q$ is the only other exterior point of $A$, $c(p)$ can have at most two exterior points and $p$ must join one of them but no other point. By the induction hypothesis, $c(p)$ is a simple path. Since $p$ is joined solely to one of the end points of this path, $A$ is also a simple path.

THEOREM 5. *Let $A$ be a $T$-connected graph. If any interior point of $A$ transposes to a non-end point of $T(A)$, then every non-end point of $A$ transposes to an exterior point of $T(A)$.*

PROOF. Let $p$ be an interior point of $A$ which is not an end point in $T(A)$, and let $q$ be a non-end point of $A$. Since interior points transpose to exterior points by Theorem 3, we can assume $q \neq p$. Denote the components in $A$ of $c(p)$ by $A_1, A_2, \cdots, A_k$, where $q$ is in $A_1$. Because $q$ has at least degree 2 in $A$, it is joined to at least one point $t$ in $A$. Let $r$ and $s$ be any two vertices in $c(p, q)$. Then, in $T(A)$, rts is a path if $r, s \in a_i$, $i \neq 1$, while $r$ and $s$ are joined if $r \in a_i$, $s \in A_j$, $i \neq j$. Finally, if $r, s \in A_1$ then rms is a path in $T(A)$, where $m$ is any point in $A$. Thus, $c(p, q)$ is connected in $T(A)$. Because $p$ is not an end point in $T(A)$, it joins some point in $T(A)$ other than $q$. Therefore, $c(q)$ is connected in $T(A)$ and $q$ is an exterior point of $T(A)$.

COROLLARY. *In a $T$-connected graph, if one non-end point transposes to an interior point, then every interior point transposes to an end point.*

THEOREM 6. *A $T$-connected graph $A$ of order $n \geqq 5$ has a $T$-connected subgraph of order $n-1$.*

PROOF. Assume that no $(n-1)$-order subgraph of $A$ is $T$-connected. Then any exterior point of $A$ transposes to an interior point of $T(A)$ so that transposition interchanges the interior and exterior sets of $A$ respectively with the exterior and interior sets of $T(A)$. If $A$ has an exterior non-end point, then from the Corollary all exterior points of $T(A)$ are end points. Thus, either all the exterior points of $A$ or all those of $T(A)$are end points, and we may suppose the former case. If $A$ had only two exterior points, then by Theorem 4 it would be a simple path. But then for $n \geqq 5$ there is a

$T$-connected subgraph of order $n-1$. There are therefore at least three points $p$, $q$, and $r$ which are end points in $A$. These form a connected triangle in $T(A)$ and each is joined to exactly $n-2$ points in $T(A)$. Let $w$ be the point not joined to $p$ in $T(A)$. Both $c(w)$ and $c(w, q)$ are connected in $T(A)$. Since $c(q)$ is connected in $A$, it is disconnected in $T(A)$ by the assumption. It follows that $q$ joins $w$ in $T(A)$. But then $c(r)$ is connected in $T(A)$, which is contradictory. The initial assumption is therefore false.

THEOREM 7. *A necessary and sufficient condition for a graph of order* $n \geqq 6$ *to be T-connected is that at least two of its* $(n-1)$*-order subgraphs be T-connected.*

PROOF. The sufficiency follows from Theorem 2. The necessity condition is valid for $n=6$ by an actual check of the possible graphs. To establish it in general, assume that it is valid for $k \geqq 6$ and consider a graph $A$ of order $k+1$ which is $T$-connected. From Theorem 6, $A$ has a $T$-connected subgraph $c(p)$ which, by assumption, has two $(k-1)$-order subgraphs which are $T$-connected. Let these be $c(p, q)$ and $c(p, r)$. If $c(q)$ is $T$-connected, the induction is established. If $c(q)$ is not $T$-connected, it may by supposed disconnected in $T(A)$. Then, because $c(p, q)$ is connected in $T(A)$, the point $p$ is not joined in $T(A)$ to any point in $c(q)$. Thus, in $A$, $p$ joins every vertex in $c(q)$. It does not join $q$ in $A$, for then $p$ would be an isolated point in $T(A)$. It follows that in both $A$ and $T(A)$ the point $p$ is joined to a vertex of $c(p, r)$. Because $c(p, r)$ is $T$-connected, this implies that $c(r)$ is $T$-connected. Thus, in every case $A$ has two $T$-connected subgraphs of order $k$ and the induction is complete.

A graph is cyclicly connected if every pair of its vertices are in some closed circuit. H. Whitney has shown [1] that for a connected graph of order $n>2$ to be cyclicly connected, it is necessary and sufficient that it have no interior points (cut points in his terminology). In order for both $A$ and $T(A)$ to be cyclicly connected it is therefore necessary and sufficient that *all* $(n-1)$-order subgraphs be $T$-connected.

A necessary and sufficient condition that a connected graph $A$ be $T$-connected is that $A$ have no partition into parts $A_1$ and $A_2$ such that every point in $A_1$ joins every point in $A_2$. In order to test a given connected graph $A$ for $T$-connectivity, choose any point $p_1$ and put it in $A_1$. Put every point not joined by $p_1$ into $A_1$. Let $p_k$ be a point joined by $p_1$. If it joins every point already in $A_1$, put it in $A_2$; otherwise put it in $A_1$. Then repeat with $p_k$: put every point not joined by $p_k$ into the component of $p_k$, and continue in this way. If in the end $A_2$ is empty, then $A$ is $T$-connected.

[1] H. Whitney, "Non-separable and planar graphs", Trans. American Math. Soc., vol. 34 (1932) pp. 339-362.

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## LETTERS FROM SUBSCRIBERS

I wish to congratulate you for the excellent job you have done, managing MATHEMATICS MAGAZINE. I wish to say that the magazine is not just a magazine printing mathematical ideas. It has an artistic value. The ideas coming from the heart of mathematicians and going into the heart are in it. I wish you much more success and I hope and pray that the magazine shall be coming out for ever and as it has been doing becomes better than ever. . . .

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Ali R. Amir-Moéz

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. . .

I wish to take this opportunity to tell you that the MATHEMATICS MAGAZINE is getting better with every issue. I like it very much and recommend it to mathematically interested friends.

I wish you would reduce the space given to very elementary as well as to very advanced subjects and instead emphasize more papers for those who are like me not professional mathematicians, but intensely interested in developments in mathematics and their implications for modern science and technology.

---

Dr. Francis Joseph Weiss

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*Comments and suggestions are welcome, and will be published from time to time.—Ed.*

# TRANSFORMATIONS OF A CONIC INTO ITSELF

G. H. Lundberg

*Foreword*. It is well known that there is a close relationship between the theory of projective transformations and the modern theory of conic sections. Indeed, it has become customary to define the conic by means of projective transformations, and many projectivities are best studied in terms of the transformations they induce on a conic. In this paper, a particular class of projective transformations, the cyclic or periodic projectivities are singled out, and the transformations they induce on the conic studied by the powerful method of generalized homogeneous coordinates. The cyclic projectivities of orders two, three and four on the conic are completely determined. In section 4, it is shown that cyclic projectivities of any order on the conic can be found.

Consider a point $A$ in a projective plane. Suppose that by $n$ successive applications of a projective transformation, $A$ is mapped back onto itself. Then the transformation is called a *cyclic*, or *periodic*, projectivity. See, for instance, Doehlemann, *Geometrische Transformationen* (W. de Gruyter, 1930), or Coxeter, *The Real Projective Plane* (McGraw-Hill, 1949).

The number $n$ is called the *order* of the projectivity. When $n$ equals two, the transformation is called an *involution*. The successive images $A = A_1, B_1, C_1, \cdots$ define a range of points which corresponds, under the transformation, to a range $A_2, B_2, C_2 \cdots$ where $A_1 = B_2$, $A_2 = B_3$, etc. This paper will discuss the conditions under which such ranges of points can be picked out on a conic. For the particular cases of $n$ equals two, three and four, it will be shown how these ranges completely determine the projectivity. It should be noticed that the standard literature usually confines itself to involutions, i.e., the case $n$ equals two.

## 1. *The cyclic projectivity of order two*

If the points $(1, 0, 0)$ and $(0, 1, 0)$ of the triangle of reference are on the conic, and if the sides opposite these points are tangent to it, the general equation of the conic,

$$ax_1^2 + 2bx_1x_2 + ex_2^2 + 2gx_1x_3 + 2fx_2x_3 + cx_3^2 = 0,$$

reduces to

$$cx_3^2 + 2bx_1x_2 = 0$$

Also if the point (1, 1, 1) is assumed to lie on the conic, the latter equation becomes

$$x_3^2 - x_1 x_2 = 0$$

without loss of generality.

In Figure 1 the points $A_1$ and $B_1$ and their homologous points $A_2$ and $B_2$ determine an involution of points on a conic. The third vertex (0, 0, 1)

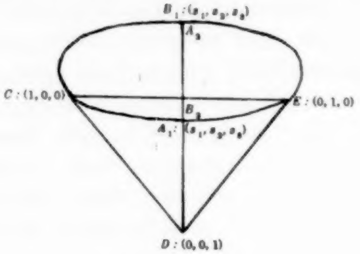


**Figure 1**

of the triangle coincides with the center of projectivity at $D$.

However, if the coordinates of $A_1$ are taken to be $(z_1, z_2, z_3)$, the equation of the straight line $DB$ is

$$x_1 z_2 - x_2 z_1 = 0$$

If this equation is solved simultaneously with the equation of the conic.

$$x_3^2 - x_1 x_2 = 0$$

the coordinates of point $B_1$ are found to be $(z_1, z_2, -z_3)$. Hence the linear transformation which carries $A_1$ into $A_2$ is

$$\rho x_1' = x_1$$

$$\rho x_2' = x_2$$

$$\rho x_3' = -x_3$$

which is order two, since a second application sends $B_1$ into $B_2$ and produces the identity. Since points $C$ and $E$ are the double points of the involution, the transformation is hyperbolic.

Transformations of parabolic and elliptic cyclic projectivities of order two may be found by placing the center of projectivity respectively on and inside the conic. The first has one double point while the second has none. All cyclic projectivities of points on a conic beyond the second

order have no double points and are therefore elliptic.

*2. The cyclic projectivity of order three*

If all the vertices of a triangle of reference are on the conic, the general equation of the conic reduces to

$$2hx_1x_2 + 2gx_1x_3 + 2fx_2x_3 = 0$$

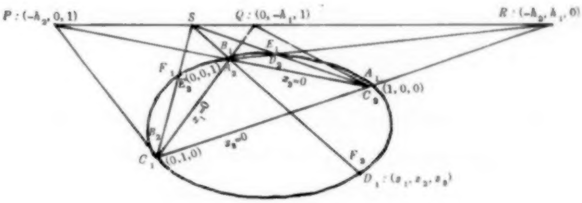

Figure 2

Since there are essentially two constants only, the equations may be further reduced to

$$x_1x_2 + h_1x_1x_3 + h_2x_2x_3 = 0$$

In Figure 2, $A_1$, $B_1$, and $C_1$ are the vertices of the reference triangle. Their homologous points $A_2$, $B_2$, and $C_2$ are arranged so that $A_2$ coincides with $B_1$, $B_2$ with $C_1$, and $C_2$ with $A_1$, so that there is a cyclic relation among them. The axis of projectivity or the Pascal line will pass through the points $P$, $Q$, and $R$ which are the intersections of the line $B_1C_2$ and the tangent at $C_1$, the line $A_2C_1$ and the tangent at $A_1$, and the line $A_1B_2$ and the tangent at $B_1$, respectively.

Since the reference triangle is in the position given, the equations for the tangents at $A_1$, $B_1$, and $C_1$ are in the order given:

$$x_2 + h_1x_3 = 0$$

$$h_1x_1 + h_2x_2 = 0$$

$$x_1 + h_2x_3 = 0$$

The coordinates of point $P$ are determined by solving simultaneously the equations of the tangent at $C_1$,

$$x_1 + h_2x_3 = 0$$

and the line $A_1B_1$,

$$x_2 = 0$$

which gives

$$P : (-h_2, 0, 1)$$

Similarly, the coordinates of $Q$ and $R$ are respectively,

$$Q : (0, -h_1, 1) \quad \text{and} \quad R : (-h_2, h_1, 0)$$

The Pascal line, determined from points $Q$ and $R$, is

$$x_2h_2 + x_1h_1 + h_1h_2x_3 = 0$$

Now $D_1$ is taken as any other point on the range $A_1$, $B_1$, $C_1$, $\cdots$ on the conic. The point homologous to $D_1$ is found by first locating point $S$, the intersection of line $A_2D_1$ with the Pascal line, and next joining $S$ with $A_1$, cutting the conic at $D_2$, the required point.

It is assumed that $D_2$ of the second range is $E_1$ of the first range. Now since both pairs of points $D_2$, $E_1$ and $C_2$, $A_1$ coincide, the lines $A_1D_2$ and $C_2E_1$ at $S$ will likewise coincide. Then the second intersection of the conic with the line $C_1S$ will locate $E_2$, the homologous point of $E_1$.

Next, if it is assumed that $E_2$ of the second range coincides with $F_1$ of the first range, and since $C_1$ and $B_2$ coincide, then line $B_2F_1$ coincides with line $C_1E_2$ which passes through point $S$. Therefore $D_1$ and $F_2$ coincide, for $B_1F_2$ and $B_2F_1$ meet at $S$.

From this, evidently, any one triad of homologous points are related in such a manner that they are the vertices of an inscribed triangle, then any other set of three homologous points will likewise be so related and will be the vertices of a second inscribed triangle. Since such sets of points are unlimited in number, the points on the conic can be regarded as grouped sets of three which are cyclically related and this relationship can be expressed as a linear transformation.

To find the third order transformation, the coordinates of $D_1$ may be expressed as $(z_1, z_2, z_3)$. Then line $A_2D_1$ is

$$z_2x_1 - z_1x_2 = 0$$

Solving $A_2D_1$ simultaneously with the Pascal line gives

$$\left(z_1,\ z_2,\ -\frac{z_1h_1 + h_2z_2}{h_1h_2}\right)$$

for the coordinates of $S$. Then line $SA_1$ is

$$(z_1h_1 + z_2h_2)x_2 + z_2h_1h_2x_3 = 0$$

The line $SA_1$ solved simultaneously with the conic,

$$x_1x_2 + h_1x_1x_3 + h_2x_2x_3 = 0$$

will give as the coordinates of point $D_2$ or $E_1$,

$$[-z_2h_2^2(z_1h_1 + z_2h_2),\ z_1z_2h_1^2h_2,\ -z_1h_1(z_1h_1 + z_2h_2)]$$

Since the point $(z_1, z_2, z_3)$ was assumed to lie on the conic,

$$x_1x_2 + h_1x_1x_3 + h_2x_2x_3 = 0$$

then

$$z_1z_2 = -z_3(h_1z_1 + h_2z_2)$$

Substituting this value of $z_1z_2$ in the coordinate values of $D_2$ or $E_1$ gives the simplified result

$$(z_2h_2^2,\ z_3h_1^2h_2,\ z_1h_1)$$

Similarly, the coordinates of $F_1$ or $E_2$ are found to be

$$(z_3h_1h_2^2,\ z_1h_1^2,\ z_2h_2)$$

From the coordinates of $D_2$ or $E_1$, the transformation carrying $D_1$ into $D_2$ is

$$T': \quad \begin{aligned} \rho x_1' &= x_2h_2^2 \\ \rho x_2' &= x_3h_1^2h_2 \\ \rho x_3' &= x_1h_1 \end{aligned}$$

This transformation is of the third order, because one repetition gives transformation

$$T'': \quad \begin{aligned} \rho x_1'' &= x_3h_1^2h_2^3 \\ \rho x_2'' &= x_1h_1^3h_2 \\ \rho x_3'' &= x_2h_1h_2^2 \end{aligned}$$

which carries $E_1$ into $E_2$, and another repetition gives

$$T''': \quad \begin{aligned} \rho x_1''' &= x_1h_1^3h_2^3 \\ \rho x_2''' &= x_2h_1^3h_2^3 \\ \rho x_3''' &= x_3h_1^3h_2^3 \end{aligned}$$

which sends $F_1$ into $F_2$. So point $F_2$ becomes the identity or point $D_1$ with coordinates $(z_1, z_2, z_3)$ — thus completing the circuit of the homologous triad of points.

### 3. *The cyclic projectivity of order four*

If the four points, $A_1$, $B_1$, $C_1$, and $D_1$, and the homologous points, $A_2$, $B_2$, $C_2$, and $D_2$, are on a conic and are cyclically related in a projectivity, then $A_2$ coincides with $B_1$; $B_2$ with $C_1$; $C_2$ with $D_1$; and $D_2$ with $A_1$. The choice of these points cannot be arbitrary for they must exclude all triads of points with their homologous points which set up a projectivity of the third order. The restriction which is placed on the four points and their homologous points in the projectivity will now be explained.

The pairs of lines $(A_1D_2, A_2D_1)$ and $(B_1C_2, B_2C_1)$ of Figure 3 are constructed so that they intersect on the axis of projectivity.

It is observed that $A_1D_2$ and $B_2C_1$ are tangents at $A_1$ and $C_1$ respectively, also that $A_2D_1$ and $B_1C_2$ are the same line. Then the two tangents and the line $B_1D_1$ have the common intersection on the axis of projectivity at the pole of $A_1C_1$. Consequently, $A_1C_1$ and $B_1D_1$ are conjugate lines and the points $A_1$, $B_1$, $C_1$ and $D_1$ are harmonic points, conjugate in pairs.

Now if the two vertices (1, 0, 0) and (0, 1, 0) of the reference triangle are $A_1$ and $C_1$, respectively, with the third vertex at the pole of $A_1C_1$, the form of the equation of the conic can be taken as

$$x_3^2 - x_1x_2 = 0$$

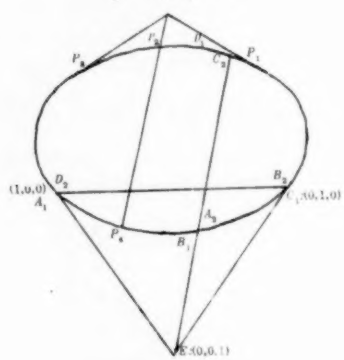


Figure 3

which was the form used for a similar situation in section 1.

If the coordinates of $B_1$ are chosen as (1, 1, 1), the line $EB_1$ is

$$x_2 - x_1 = 0$$

Solving the line simultaneously with the conic,

$$x_3^2 - x_1 x_2 = 0$$

point $D_1$ is found to be $(1, 1, -1)$. Then the transformation which carries $A_1$ into $A_2$, $B_1$ into $B_2$, $C_1$ into $C_2$, and finally $D_2$ into $A_1$ is

$$\rho x_1' = x_1 + x_2 - 2x_3$$

$$\rho x_2' = x_1 + x_2 + 2x_3$$

$$\rho x_3' = x_1 - x_2$$

Now if any point on the conic such as $P_1$ with coordinates $(z_1, z_2, \sqrt{z_1 z_2})$ is considered, the above transformation will carry such point into its homologous point $P_2$. Thus

$$\rho x_1' = x_1 + x_2 - 2\sqrt{x_1 x_2}$$

$$\rho x_2' = x_1 + x_2 + 2\sqrt{x_1 x_2}$$

$$\rho x_3' = x_1 - x_2$$

So the coordinates of $P_2$ are

$$[(\sqrt{z_1} - \sqrt{z_2})^2, (\sqrt{z_1} + \sqrt{z_2})^2, (z_1 - z_2)]$$

Repeating the transformation gives

$$\rho x_1'' = x_2$$

$$\rho x_2'' = x_1$$

$$\rho x_3'' = -\sqrt{x_1 x_2}$$

which carries $P_2$ into the conjugate of $P_1$ or $P_3$ whose coordinates are $(z_2, z_1, -\sqrt{z_1 z_2})$. If $P_3$ is carried into $P_4$, the conjugate of $P_2$, by a third application of the transformation, the result is

$$\rho x_1''' = x_2 + x_1 + 2\sqrt{x_1 x_2}$$

$$\rho x_2''' = x_2 + x_1 - 2\sqrt{x_1 x_2}$$

$$\rho x_3''' = x_2 - x_1$$

Then the coordinates of $P_4$ are

$$[(\sqrt{z_1}+\sqrt{z_2})^2,\ (\sqrt{z_1}-\sqrt{z_2})^2,\ (z_2-z_1)]$$

Finally another application of the transformation gives

$$\rho x_1^{iv} = x_1$$

$$\rho x_2^{iv} = x_2$$

$$\rho x_3^{iv} = \sqrt{x_1 x_2}$$

which completes the circuit since $P_4$ is carried into $P_1$, the initial point.

Although it has been demonstrated that the transformation is of the fourth order for any point on the conic, it is further necessary to show that certain sets of four points are harmonically related. The equation of any tangent line to the given conic is

$$(x_1 - x_1')(-x_2') + (x_2 - x_2')(-x_1') + (x_3 - x_3')(2x_3') = 0$$

So the tangents at $P_1$ and $P_2$ are respectively

$$z_2 x_1 + z_1 x_2 - 2\sqrt{z_1 z_2 x_3} = 0$$

and

$$z_1 x_1 + z_2 x_2 + 2\sqrt{z_1 z_2 x_3} = 0$$

Solving these equations simultaneously gives

$$x_1 : x_2 : x_3 = \begin{vmatrix} z_1 - 2\sqrt{z_1 z_2} \\ z_2 & 2\sqrt{z_1 z_2} \end{vmatrix} : - \begin{vmatrix} z_2 - 2\sqrt{z_1 z_2} \\ z_1 & 2\sqrt{z_1 z_2} \end{vmatrix} : \begin{vmatrix} z_2 & z_1 \\ z_1 & z_2 \end{vmatrix}$$

So the coordinates of the point of intersection of the tangents are

$$[-2\sqrt{z_1 z_2},\ 2\sqrt{z_1 z_2},\ (z_1 - z_2)]$$

The determinant of the coordinate values of points $P_2$, $P_4$, and the intersection of the tangents is

$$\begin{vmatrix} -2\sqrt{z_1 z_2} & (\sqrt{z_1}-\sqrt{z_2})^2 & (\sqrt{z_1}+\sqrt{z_2})^2 \\ 2\sqrt{z_1 z_2} & (\sqrt{z_1}+\sqrt{z_2})^2 & (\sqrt{z_1}-\sqrt{z_2})^2 \\ (z_1 - z_2) & (z_1 - z_2) & (z_2 - z_1) \end{vmatrix}$$

which is equal to zero; so the three points are linearly dependent and hence lie on a line.

Consequently the points $P_1$, $P_2$, $P_3$, and $P_4$ are harmonically separated

by the conjugate lines $P_1P_3$ and $P_2P_4$ because the tangents at $P_1$ and $P_3$ together with the line $P_2P_4$ intersect.

Therefore since the transformation carries any four harmonic points $P_1$, $P_2$, $P_3$, and $P_4$ successively into each other, it is of a cyclic projectivity of order four.

4. *Cyclic projectivities of any order*

To find the cyclic projectivities of any order, a circle of radius $\frac{\sqrt{2}}{2}$, in the metric plane is first considered. Let the triangle of reference be situated so that its sides in terms of Cartesian coordinates are

$$\begin{aligned} x + y - 1 &= 0 \\ x - y - 1 &= 0 \\ 2x \quad\ - 1 &= 0 \end{aligned}$$

These equations in terms of homogeneous Cartesian coordinates are

$$\begin{aligned} x + y - z &= 0 \\ x - y - z &= 0 \\ 2x \quad\ - z &= 0 \end{aligned}$$

Then the transformation from homogeneous Cartesian coordinates to homogeneous projective coordinates is

$$\begin{aligned} \rho x_1 &= K_1(x + y - z) \\ \rho x_2 &= K_2(x - y - z) \\ \rho x_3 &= K_3(2x - z) \end{aligned}$$

Solving these equations simultaneously for $x$, $y$, and $z$ gives the inverse transformation,

$$x = -\frac{\rho x_1}{2K_1} - \frac{\rho x_2}{2K_2} + \frac{\rho x_3}{K_3}$$

$$y = \frac{\rho x_1}{2K_1} - \frac{\rho x_2}{2K_2}$$

$$z = \frac{\rho x_1}{K_1} - \frac{\rho x_2}{K_2} + \frac{\rho x_3}{K_3}$$

Substituting these values in the equation of the general conic,

$$ax_1^2 + 2bx_1x_2 + ex_2^2 + 2gx_1x_3 + 2fx_2x_3 + cx_3^2 = 0$$

gives

$$\frac{\rho}{K_3^2}x_3^2 - \frac{2\rho}{K_1K_2}x_1x_2 = 0$$

For convenience it is assumed that $\rho = 2$ and $K_1 = K_2 = K_3 = 1$. Then

$$x_3^2 - 2x_1x_2 = 0$$

which is the equation of the conic into which the original circle in terms of homogeneous Cartesian coordinates, namely,

$$x^2 + y^2 - \frac{1}{2}z^2 = 0$$

is projected.

Using the same convenient values for the $K$'s and $\rho$, the transformation for carrying points on the conic in the projective plane into points on the circle in the projective plane is

$$D: \begin{aligned} x &= -x_1 - x_2 + 2x_3 \\ y &= x_1 - x_2 \\ z &= -2x_1 - 2x_2 + 2x_3 \end{aligned}$$

of which the inverse is

$$D^{-1}: \begin{aligned} x_1 &= \frac{x}{2} + \frac{y}{2} - \frac{z}{2} \\ x_2 &= \frac{x}{2} - \frac{y}{2} - \frac{z}{2} \\ x_3 &= x \qquad - \frac{z}{2} \end{aligned}$$

A second order transformation for an elliptic involution in terms of homogeneous Cartesian coordinates is

$$\begin{aligned} x' &= -x \\ y' &= -y \\ z' &= z \end{aligned}$$

whose inverse is

$$T_2: \begin{aligned} x &= -x' \\ y &= -y' \\ z &= z' \end{aligned}$$

which is likewise of the second order. Now if the matrix of transformation $T_2$ is transformed by the matrix of transformation $D$, the result is a transformation which carries points into points in terms of homogeneous pro-

jective coordinates. Thus

$$(D^{-1}T_2D) \text{ or } \left\|\begin{matrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{2} \end{matrix}\right\| \cdot \left\|\begin{matrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} -1 & -1 & 2 \\ 1 & -1 & 0 \\ -2 & -2 & 2 \end{matrix}\right\| = \left\|\begin{matrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{matrix}\right\|$$

which is the matrix of the second order transformation,

$$\begin{aligned} x_1' &= x_1 + 2x_2 - 2x_3 \\ x_2' &= 2x_1 + x_2 - 2x_3 \\ x_3' &= 2x_1 + 2x_2 - 3x_3 \end{aligned}$$

Likewise if the inverse second order transformation of a hyperbolic involution is

$$T': \begin{aligned} x &= -x' \\ y &= \phantom{-}y' \\ z &= \phantom{-}z' \end{aligned}$$

Then

$$(D^{-1}T'D) = \left\|\begin{matrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{2} \end{matrix}\right\| \cdot \left\|\begin{matrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} -1 & -1 & 2 \\ 1 & -1 & 0 \\ -2 & -2 & 2 \end{matrix}\right\|$$

gives the matrix of the second order transformation,

$$\begin{aligned} x_1' &= 2x_1 + x_2 - 2x_3 \\ x_2' &= x_1 + 2x_2 - 2x_3 \\ x_3' &= 2x_1 + 2x_2 - 3x_3 \end{aligned}$$

The third order transformation in terms of homogeneous Cartesian coordinates is found by letting $\theta = 120°$ in the transformation of rotation,

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z \end{aligned}$$

and then finding the inverse. This gives transformation

$$T_3: \begin{aligned} x &= -\frac{1x'}{2} + \frac{\sqrt{3}y'}{2} \\ y &= -\frac{\sqrt{3}x'}{2} - \frac{y'}{2} \\ z &= z' \end{aligned}$$

Now if $T_3$ is transformed by $D$, the result is the third order transformation in terms of homogeneous projective coordinates,

$$x_1' = \frac{(2+\sqrt{3})\,x_1}{2} + \frac{3\,x_2}{2} - \frac{(3+\sqrt{3})\,x_3}{2}$$

$$x_2' = \frac{3\,x_1}{2} + \frac{(2-\sqrt{3})\,x_2}{2} - \frac{(3-\sqrt{3})x_3}{2}$$

$$x_3' = \frac{(3+\sqrt{3})\,x_1}{2} + \frac{(3-\sqrt{3})\,x_2}{2} - 2x_3$$

The inverse fourth order transformation in terms of homogeneous Cartesian coordinates of the like order transformation found by letting $\theta = 90°$ in the equations of rotation is

$$T_4: \begin{aligned} x &= y' \\ y &= -x' \\ z &= z' \end{aligned}$$

Then $(D^{-1}T_4D)$ gives the fourth order transformation in terms of homogeneous projective coordinates,

$$\begin{aligned} x' &= x_1 \\ x_2' &= 2x_1 + x_2 - 2x_3 \\ x_3' &= 2x_1 \qquad - x_3 \end{aligned}$$

If the rotation is through any angle, $\frac{2\pi K}{n}$, the inverse of the resulting transformation is

$$T_n: \begin{aligned} x &= \cos\frac{2\pi K}{n}x' + \sin\frac{2\pi K}{n}y' \\ y &= -\sin\frac{2\pi K}{n}x' + \cos\frac{2\pi K}{n}y' \\ z &= z' \end{aligned}$$

The square of the matrix of this transformation is

$$\left\| \begin{matrix} \cos\frac{4\pi K}{n} & \sin\frac{4\pi K}{n} & 0 \\ -\sin\frac{4\pi K}{n} & \cos\frac{4\pi K}{n} & 0 \\ 0 & 0 & 1 \end{matrix} \right\|$$

which is a matrix where all the angles involved in the trigonometric functions are twice those of the original matrix.

Therefore by mathematical induction it may be proved that the $n$-th power of the matrix $T_n$ gives

$$\left\| \begin{matrix} \cos 2\pi K & \sin 2\pi K & 0 \\ -\sin 2\pi K & \cos 2\pi K & 0 \\ 0 & 0 & 1 \end{matrix} \right\|$$

which is the identity matrix, of which the corresponding transformation is

$$x_1' = x_1$$
$$x_2' = x_2$$
$$x_3' = x_3$$

Hence transformation $T_n$ is of order $n$.

Now the order of any non-singular square matrix is the same as the order of its transformation by another non-singular square matrix. This was previously verified when each transformation, $T_2$, $T_2'$, $T_3$, and $T_4$, was found to be of the same order as its respective transformation by $D$. Hence the transformation of $T_n$ by $D$ is of order $n$. Since $(D^{-1}T_nD)^n$ gives the identity, transformations of any order in homogeneous projective coordinates can be found and thus projectivities of all orders can be constructed upon the conic.

## *BIBLIOGRAPHY*

Coxeter, Harold. *The Real Projective Plane.* New York, McGraw-Hill Book Co., Inc., 1949.

Doehlemann, Karl. *Geometrische Transformationen.* Leipzig, W. de Gruyter and Co., 1930.

Patterson, Boyd. *Projective Geometry.* New York, John Wiley and Sons, Inc., 1937.

Veblen, Oswald and Young, John. *Projective Geometry.* New York, Ginn and Co., 1910, 1918.

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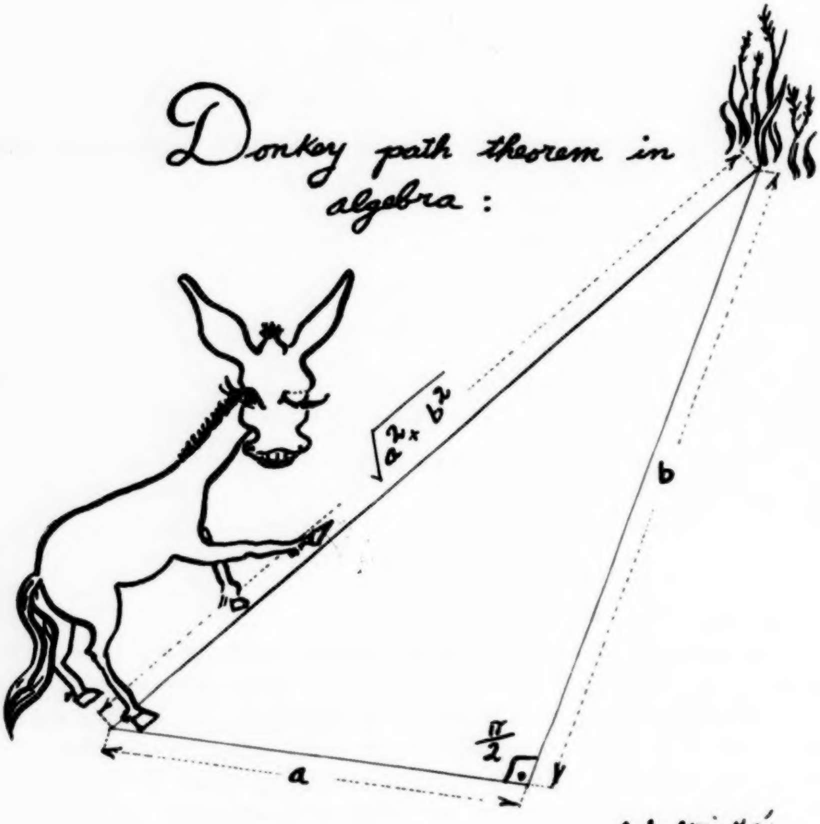
# Donkey path theorem in algebra:




A. R. Amir-Moéz

If $a \neq 0$ and $b \neq 0$,

then $\sqrt{a^2+b^2} \neq a+b$, and in fact

$$\sqrt{a^2+b^2} < a+b.$$

*Edited by Joseph Seidlin and C. N. Shuster*

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, as a teacher, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

# TEACHING TRIGONOMETRY THROUGH VECTORS

Ali R. Amir-Moéz

It takes a long time for any relatively new idea to move into the classroom. I am sure that our students in high school as well as in college would not have any trouble with the following approach to trigonometry. This approach gets us through faster.

1. DEFINITION: A vector is a directed line segment. For example the vector $AB$, denoted by $\vec{AB}$, is the line segment $AB$ with a sense from $A$ to $B$. $A$ is called the beginning of $\vec{AB}$. Throughout this note, we consider only vectors with beginning at the origin 0 of a rectangular Cartesian co-ordinate system and we denote $\vec{0A}$ by $\vec{A}$. The co-ordinates of the point $A$ are called the components of $\vec{A}$. In particular we call the abscissa of $A$ the $x$-component of $\vec{A}$, the ordinate of $A$ the $y$-component of $\vec{A}$ and we call them respectively $x$ and $y$.

2. TRIGONOMETRIC FUNCTIONS OF AN ANGLE: It can be explained, intuitively, that for any angle $\theta$ which is measured counter clock wise, we can put the vertex at the origin 0 and the first side on $x$-axis. Choose a vector $\vec{A}$ of unit length on the second, or terminal side of $\theta$ (Fig. 1). Then

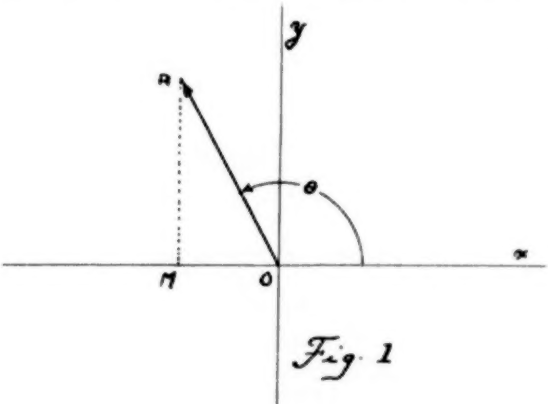



Fig. 1

we define the $x$-component of $\vec{A}$ to be $\cos\theta$, and the $y$-component of $\vec{A}$ to be $\sin\theta$. We also define $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\operatorname{ctn}\theta = \frac{\cos\theta}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, and $\csc\theta = \frac{1}{\sin\theta}$. We call $\sin\theta$, $\cos\theta$, $\tan\theta$, $\operatorname{ctn}\theta$, $\sec\theta$, and $\csc\theta$ the trigonometric functions of $\theta$. Right away we can give relations between the

trigonometric functions of any angle and an angle terminating in the first quadrant. The application of trigonometric functions to solving a right triangle and a triangle in general can be given here.

Now we should stop a few minutes and give a brief review of absolute value of a number. We can always construct a right triangle whose hypotenuse is $0A$ and the other two sides of it are absolute values of $\sin\theta$ and $\cos\theta$ and as we know they are denoted by $|\sin\theta|$ and $|\cos\theta|$ (Fig. 1). Squaring a number we get the same as we square its absolute value. Therefore,

$$\cos^2\theta + \sin^2\theta = 1,$$

for any $\theta$. Here other relations can be discussed. Indeed here is the place to talk about some trigonometric equations.

3. THE ADDITION FORMULAS. To avoid use of complicated diagrams we develop a few ideas about vectors.

3.1 ADDITION OF VECTORS: Given two vectors $\vec{A}$ and $\vec{B}$ we can construct a unique parallelogram $0ARB$ (Fig. 2). We define $\vec{R}$ to be $\vec{A}+\vec{B}$.

3.2 INNER PRODUCT OF TWO VECTORS: Let $\vec{A}$ and $\vec{B}$ be two vectors. Let the angle between these vectors be $\alpha$. We define the inner product

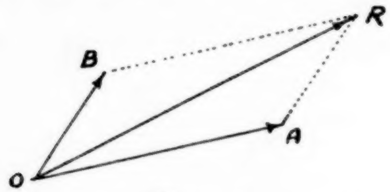



Fig. 2

of $\vec{A}$ and $\vec{B}$ to be

$$\vec{A}\cdot\vec{B} = |\vec{A}|\cdot|\vec{B}|\cos\alpha,$$

where $|\vec{A}|$ means the length of $\vec{A}$ and $|\vec{B}|$ means the length of $\vec{B}$.

Professor G. B. Price of University of Kansas said, "Suppose you tell your students that we study trigonometry to find out how high a tree is. Many of them will answer; so what!" I certainly agree with him and believe that using vectors brings new ideas and new interests. In fact I believe it makes the subject much easier.

3.3 PROJECTION OF A VECTOR ON AN AXIS: Let $0p$ be an axis through 0, not necessarily the same as $x$-axis or $y$-axis. The projection of $\vec{A}$ on $0p$ is the signed length of $0P$, denoted by $0P$, where $P$ is the foot of the perpendicular through $A$ to $0p$ (Fig. 3). If $\vec{A}$ makes an angle $\alpha$ with

the positive direction of $0p$, then $0P = |\vec{A}| \cos \alpha$.

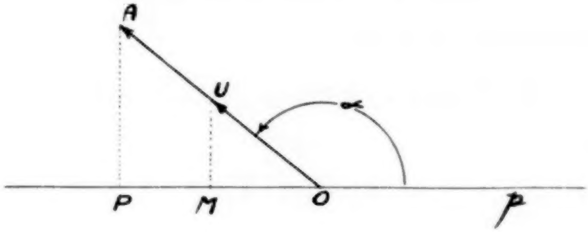



Fig. 3

Proof: Choose a unit vector $\vec{U}$ on $\vec{0A}$. By definition 2, the projection of $\vec{U}$ on $0p$ is $\cos \alpha$. Using similar triangles $0UM$ and $0AP$ the proof is clear. Note that the proof is independent of $\alpha$ being less than $\frac{\pi}{2}$.

3.4 THEOREM: The projection of $\vec{R} = \vec{A} + \vec{B}$ on $0p$ is equal to the sum of the projections of $\vec{A}$ and of $\vec{B}$ on $0p$.

Indeed the proof is quite simple and I don't think I should bother you with it. Only I refer you to (Fig. 4).

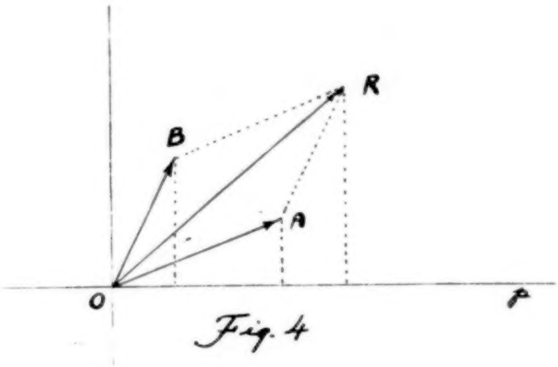



Fig. 4

3.5 THEOREM: The inner multiplication follows the laws of ordinary multiplication. For example:

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}.$$

Proof: Suppose that the angle between $\vec{B}$ and $\vec{A}$ is $\alpha$, the angle between $\vec{C}$ and $\vec{A}$ is $\beta$, and the angle between $\vec{R} = \vec{B} + \vec{C}$ and $\vec{A}$ is $\gamma$ (Fig. 5).

By 3.4

$$|\vec{R}|\cos\gamma = |\vec{B}|\cos\alpha + |\vec{C}|\cos\beta.$$

Multiplying through by $|\vec{A}|$ we get

$$|\vec{A}|\cdot|\vec{R}|\cos\gamma = |\vec{A}||\vec{B}|\cos\alpha + |\vec{A}||\vec{C}|\cos\beta,$$

which means

$$\vec{A}\cdot\vec{R} = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}.$$

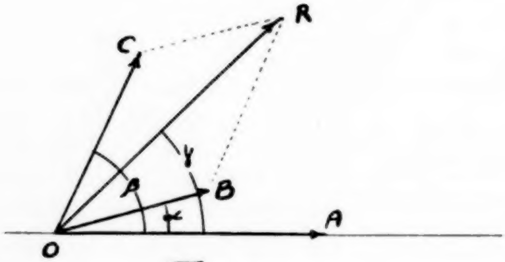



*Fig. 5*

3.6 THEOREM: If the components of $\vec{A}$ are $x_1$, $y_1$, and the components of $\vec{B}$ are $x_2$, $y_2$, then $\vec{A}\cdot\vec{B} = x_1x_2 + y_1y_2$.

Proof: Let $\vec{X}_1$ and $\vec{X}_2$ be two vectors on the $x$-axis such that $|\vec{X}_1| = x_1$ and $|\vec{X}_2| = x_2$. Let $\vec{Y}_1$ and $\vec{Y}_2$ be two vectors on the $y$-axis such that $|\vec{Y}_1| = y_1$ and $|\vec{Y}_2| = y_2$. Therefore $\vec{A} = \vec{X}_1 + \vec{Y}_1$ and $\vec{B} = \vec{X}_2 + \vec{Y}_2$. Then by 3.5 we have

$$\vec{A}\cdot\vec{B} = (\vec{X}_1 + \vec{Y}_1)\cdot(\vec{X}_2 + \vec{Y}_2) = \vec{X}_1\cdot\vec{X}_2 + \vec{X}_1\cdot\vec{Y}_2 + \vec{Y}_1\cdot\vec{X}_2 + \vec{Y}_1\cdot\vec{Y}_2.$$

But by 3.2

$$\vec{X}_1\cdot\vec{X}_2 = x_1x_2,\ \vec{X}_1\cdot\vec{Y}_2 = \vec{Y}_1\cdot\vec{X}_2 = 0, \text{ and}$$

$$\vec{Y}_1\cdot\vec{Y}_2 = y_1y_2.$$

Consequently

$$\vec{A}\cdot\vec{B} = x_1x_2 + y_1y_2.$$

3.7 THEOREM: For any two angles $\alpha$ and $\beta$ we have

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Proof: Suppose $\vec{A}$ is a vector of unit length on the terminal side of $\alpha$ and $\vec{B}$ a vector of unit length on the terminal side of $\beta$ (Fig. 6). The diagram

is not really necessary. Let $x_1$, $y_1$ be the components of $\vec{A}$ and $x_2$, $y_2$ be

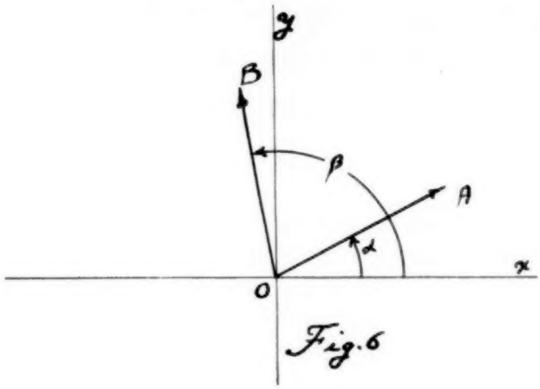



Fig. 6

the components of $\vec{B}$. By 2. we know $x_1 = \cos\alpha$, $y_1 = \sin\alpha$, $x_2 = \cos\beta$ and $y_2 = \sin\beta$. By 3.6

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos(\beta-\alpha) = (1)\cdot(1)\cos(\alpha-\beta) =$$

$$x_1 x_2 + y_1 y_2 = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Other formulas can be deduced from what has been said.

Ordinarily the proofs of the addition formulas are given through the use of a complicated diagram where $\alpha$ and $\beta$ are both less than $\frac{\pi}{2}$ and other cases are to be believed or proved by the students; that is almost never done.

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In response to a request from a teacher, we are preparing enlargements of the Tree on the cover of the book "The Tree of Mathematics". These enlargements will be four times the size of the one on the inside back cover of this magazine. They can be secured by sending 75 cents to the *DIGEST PRESS, Pacoima, California* or will be sent free with orders of 10 or more books, upon request.

# PARTICULAR SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

C. A. Grimm

Operational methods for finding particular solutions of linear differential equations with constant coefficients are mentioned in many differential equations text books, however the authors usually feel obliged to insert a preliminary paragraph apologizing for what they are about to do. The methods are given with the only justification of their use being that they do seem to produce correct particular solutions. The purpose of this note is to justify, in a manner sufficiently elementary to present to a beginning class, the more common operational methods. Specifically, the often used device of dividing the operator into 1; the use of the shift formula as an integral shift formula and the use of operators on sine and cosine functions. Additional topics will creep in as by products but the above outline indicates the main points. Actual solution for particular solutions usually requires a combination of the methods outlined above, therefore it will be assumed that the formal manipulations are known and this note will only be concerned with the justification of the individual methods. All results come from a well known integral operator which we now derive. The note will be limited to second order equations, the most common type.

Consider the linear equation,

$$\frac{dy}{dt} - sy = f(t), \tag{1}$$

whose solution is easily found to be, by non-operational methods,

$$y = e^{st}\int e^{-st}dt. \tag{2}$$

Defining

$$Dy = \frac{dy}{dt}$$

we have the operator form of (1)

$$(D-s)y = f(t)$$

The formal solution is then

(3) $$y = \frac{1}{D-s} f(t)$$

We then define the right side of (3) by

(4) $$\frac{1}{D-s} f(t) = e^{st} \int e^{-st} f(t) dt$$

In particular if $s = 0$,

$$\frac{1}{D} f(t) = \int f(t) dt$$

If we integrate (4) by parts with

$$u = f(t) \qquad dv = e^{-st} dt$$

$$du = f'(t) dt \qquad v = -\frac{1}{s} e^{-st}$$

we have

(5) $$\frac{1}{D-s} f(t) = -\frac{1}{s} f(t) + \frac{1}{s} \frac{1}{D-s} f'(t)$$

We then use (5) to find

$$\frac{1}{D-s} f'(t) = -\frac{1}{s} f'(t) + \frac{1}{s} \frac{1}{D-s} f''(t)$$

Hence

$$\frac{1}{D-s} f(t) = -\frac{1}{s} f(t) - \frac{1}{s^2} f'(t) + \frac{1}{s^2} \frac{1}{D-s} f''(t)$$

Continuing in this manner we find

$$\frac{1}{D-s} f(t) = -\frac{1}{s} f(t) - \frac{1}{s^2} f'(t) - \frac{1}{s^3} f''(t) - \cdots - \frac{1}{s^{n+1}} f^n(t) + \frac{1}{s^{n+1}} \frac{1}{D-s} f^{n+1}(t).$$

(6) $$\frac{1}{D-s} f(t) = -\frac{1}{s} \left\{ 1 + \frac{D}{s} + \frac{D^2}{s^2} + \cdots + \frac{D^n}{s^n} \right\} f(t) + \frac{1}{s^{n+1}} \frac{1}{D-s} f^{n+1}(t).$$

We define

$$-\frac{1}{s} \left\{ 1 + \frac{D}{s} + \frac{D^2}{s^2} + \cdots + \frac{D^n}{s^n} \right\} f(t)$$

as the principle part of (6). In particular if $f(t)$ is an $n$th order polynomial in $t$,

$$f(t) = a_0 + a_1 t + \cdots + a_n t^n, \qquad a_n \neq 0$$

then $f^{n+1}(t) = 0$ while all previous derivatives differ from 0, hence

(7) $$\frac{1}{s^{n+1}} \frac{1}{D-s} f^{n+1}(t) = \frac{1}{s^{n+1}} \frac{1}{D-s} \cdot 0 = \frac{1}{s^{n+1}} e^{st} \int 0 \, dt = \frac{c}{s^{n+1}} e^{st}$$

$c$ a constant. However a solution of the form obtained in (7) is already present in the solution of the reduced equation of (1). Hence to find a particular solution we need only consider the principle part of (6). This

may be obtained by formally dividing $-s+D$ into 1.

We now consider the equation

$$(D-r)(D-s)y = f(t) \qquad r \neq s. \tag{8}$$

Let

$$(D-s)y = Y$$

then

$$(D-r)Y = f(t)$$

hence

$$Y = \frac{1}{D-r}f(t)$$

and

$$y = \frac{1}{D-s}\left[\frac{1}{D-r}f(t)\right] \tag{9}$$

$$= e^{st}\int\left\{e^{(r-s)t}\int e^{-rt}f(t)dt\right\}dt. \tag{10}$$

We now integrate (10) by parts with

$$u = \int e^{-rt}f(t)dt \qquad dv = e^{(r-s)t}dt$$

$$du = e^{-rt}f(t)dt \qquad v = \frac{1}{r-s}e^{(r-s)t}$$

thus

$$y = e^{st}\left\{\frac{e^{(r-s)t}}{r-s}\int e^{-rt}f(t)dt - \frac{1}{r-s}\int e^{-st}f(t)dt\right\}$$

$$= \frac{1}{r-s}\left(\frac{1}{D-r}-\frac{1}{D-s}\right)f(t). \tag{11}$$

Formula (11) may be formally obtained from (9) by partial fractions. We may then obtain the particular solution by finding the difference of the principle parts of $\frac{1}{D-r}f(t)$ and $\frac{1}{D-s}f(t)$ and multiplying by $\frac{1}{r-s}$. Of course the polynomial in $D$ resulting from the difference of these principle parts must be the same polynomial that would be found by formally dividing $rs-(s+r)D+D^2$ into 1 and limiting the polynomial to those powers of $D$ which are necessary. If we interchange $r$, and $s$ in (8) we must also interchange them in (11), but this leaves (11) unchanged, hence the order of the operator is immaterial. For the case $r=s$ we apply the $\frac{1}{D-r}$ operator to $\frac{1}{D-r}f(t)$ but the resulting polynomial in $D$ is again the same as the one that would be obtained by formally dividing $r^2-2rD+D^2$ into 1.

We now consider the case

$$\frac{1}{D-r}\cdot\frac{1}{D-s}e^{mt} = \frac{1}{P(D)}e^{mt} \qquad m \neq r,\ m \neq s$$

$$\frac{1}{D-r}\cdot\frac{1}{D-s}e^{mt} = \frac{1}{D-r}e^{st}\int e^{(m-s)t}dt$$

$$= \frac{1}{D-r}\frac{1}{m-s}e^{mt}$$

$$= \frac{1}{m-s}e^{rt}\int e^{(m-r)t}dt$$

$$= \frac{1}{m-s}\cdot\frac{1}{m-r}e^{mt}$$

Hence

$$\frac{1}{P(D)}e^{mt} = \frac{1}{P(m)}e^{mt} \qquad P(m)\neq 0. \tag{12}$$

If the particular solution is of the form

$$\frac{1}{P(D)}e^{mt}f(t) = \frac{1}{D-r}\cdot\frac{1}{D-s}e^{mt}f(t),$$

then

$$\frac{1}{D-s}e^{mt}f(t) = e^{st}\int e^{(m-s)t}f(t)dt$$

$$= e^{mt}e^{(s-m)t}\int e^{-(s-m)t}f(t)dt$$

$$= e^{mt}\frac{1}{D-s+m}f(t).$$

Let

$$\frac{1}{D-s+m}f(t) = g(t)$$

then

$$\frac{1}{D-r}e^{mt}g(t) = e^{mt}\frac{1}{D-r+m}g(t)$$

hence

$$\frac{1}{D-r}\frac{1}{D-s}e^{mt}f(t) = e^{mt}\frac{1}{D-r+m}\frac{1}{D-s+m}f(t)$$

or

$$\frac{1}{P(D)}e^{mt}f(t) = e^{mt}\frac{1}{P(D+m)}f(t) \tag{13}$$

(13) may also be used to cover the case $P(m)=0$ excluded in the previous method. Formulas (12) and (13) are integral operator formulas that are standardly first considered from the differential operator standpoint and then used without further comment as integral operators.

Next consider the case

$$\frac{1}{P(D^2)}\begin{vmatrix}\sin mt\\ \cos mt\end{vmatrix}$$

Since $e^{imt} = \cos mt + i\sin mt$

$$\frac{1}{P(D^2)}\cos mt = R\left\{\frac{1}{P(D^2)}e^{imt}\right\}$$

$$\frac{1}{P(D^2)}\sin mt = I\left\{\frac{1}{P(D^2)}e^{imt}\right\}$$

And since

$$\frac{1}{P(D^2)}e^{imt} = \frac{1}{P[(im)^2]}e^{imt} = \frac{1}{P(-m^2)}e^{imt} \qquad \text{by (12),}$$

we have

(14) $$\frac{1}{P(D^2)}\begin{vmatrix}\sin mt\\ \cos mt\end{vmatrix} = \frac{1}{P(-m^2)}\begin{vmatrix}\sin mt\\ \cos mt\end{vmatrix} \qquad P(-m^2)\neq 0$$

For the case

(15) $$\frac{1}{D-r}\sin mt = I\left\{\frac{1}{D-r}e^{imt}\right\}$$

we have by (12)

$$\frac{1}{D-r}e^{imt} = -\frac{1}{r-im}e^{imt} = -\frac{r+im}{r^2+m^2}e^{imt}$$

$$= -\frac{r+im}{r^2+m^2}(\cos mt + i\sin mt)$$

hence

$$\frac{1}{D-r}\sin mt = -\frac{1}{r^2+m^2}\{m\cos mt + r\sin mt\}$$

$$= (D+r)\left\{-\frac{1}{r^2+m^2}\sin mt\right\}$$

$$= (D+r)\left\{\frac{1}{D^2-r^2}\sin mt\right\} \qquad \text{by (14)}$$

This, however, may formally be obtained by "multiplying" (15) in numerator and denominator by $D+r$. The case

$$\frac{1}{D^2+aD+b}\sin mt$$

may be reduced to the former case by observing by (12) that

$$\frac{1}{P(D^2)+P_1(D)}\, e^{imt} = \frac{1}{P(-m^2)+P_1(D)}\, e^{imt}.$$

The operations on cos $mt$ are entirely similar.

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## MISCELLANEOUS NOTES

*Edited by*

Charles K. Robbins

Articles intended for this department should be sent to *Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Ind.*

# AN ALGEBRAIC METHOD FOR FINDING THE CRITICAL VALUES OF THE CUBIC FUNCTION

Harry S. Clair

Central to the idea of this paper is the inequality existing between the geometric and arithmetic means of a set of positive numbers. Let $x_1, x_2, \cdots, x_n$ be such a set. Then the "theorem of the means" asserts:

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leqq \frac{x_1 + x_2 + \cdots + x_n}{n} \tag{1}$$

with equality valid only when $x_1 = x_2 = \cdots = x_n$.* For the purpose of this paper I shall need relation (1) only for the case $n = 3$, which on cubing both sides, becomes

$$x_1 x_2 x_3 \leqq \frac{(x_1 + x_2 + x_3)^3}{27} \tag{2}$$

with equality holding only when $x_1 = x_2 = x_3$. Inequality (2) may be translated into the following maximum problem: If the sum of three positive numbers is held constant, then their product will be a maximum when the three numbers are equal.

We shall now apply inequality (2) and the remark that follows to the cubic function: $y = ax^3 + bx^2 + cx + d$. It may be reduced by the transformation:

$$x = X - \frac{b}{3a}, \qquad y = Y + d - \frac{bc}{3a} + \frac{2b^3}{27a^2}$$

to the canonical form:

$$Y = a(X^3 - kX) \qquad k = \frac{b^2}{3a^2} - \frac{c}{a}. \tag{3}$$

First assume $a > 0$, $k > 0$. Then by factoring

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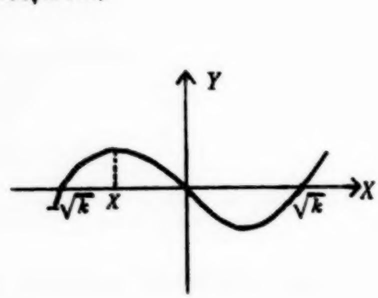
*For a proof, see G. H. Hardy, *A Course of Pure Mathematics*, Cambridge University Press, 1921, p. 32.

$$\frac{Y}{a} = (-X)(\sqrt{k}-X)(\sqrt{k}+X)$$

where the three factors on the right are each positive when $-\sqrt{k}<X<0$. Our problem is to maximize $Y/a$ with $-\sqrt{k}<X<0$. To do this, insert positive factors $\alpha$, $\beta$, $\gamma$ so that

$$\frac{Y}{a}=\frac{1}{\alpha\beta\gamma}[\alpha(-X)\beta(\sqrt{k}-X)\gamma(\sqrt{k}+X)]$$

where the sum of the factors in the brackets shall be constant, i.e., independent of $X$. This implies that




(4) $$-\alpha-\beta+\gamma = 0.$$

If now there exists a number $X$ $(-\sqrt{k}<X<0)$ such that

(5) $$-\alpha X = \beta(\sqrt{k}-X) = \gamma(\sqrt{k}+X)$$

then $X$ must locate the maximum of $Y/a$. Eliminating $X$ and $\gamma$ between equations (4) and (5), we have

(6) $$\alpha^2-2\alpha\beta-2\beta^2 = 0.$$

Assign the value $\alpha=2$. Then by (6), $\beta=\sqrt{3}-1$; and by (4), $\gamma=\sqrt{3}+1$. Solving for $X$ in equations (5) and substituting these values for $\alpha$, $\beta$, $\gamma$, we obtain

$$X = \frac{\beta}{\beta-\alpha}\sqrt{k} = -\frac{\gamma}{\alpha+\gamma}\sqrt{k} = \frac{\sqrt{3}-1}{\sqrt{3}-3}\sqrt{k} = -\frac{1}{3}\sqrt{3k}.$$

In terms of the original variable $x$, the maximum point is given by:

$$x = X-\frac{b}{3a} = -\frac{1}{3}\sqrt{\frac{b^2}{a^2}-\frac{3c}{a}}-\frac{b}{3a}=\frac{-b-\sqrt{b^2-3ac}}{3a}$$

which agrees with the result obtained by the Calculus.

To find the minimum point of $Y/a$, we consider the maximum of $-Y/a=X(\sqrt{k}-X)(\sqrt{k}+X)$, proceed as before, and arrive at the value $X=\frac{1}{3}\sqrt{3k}$. Hence $Y/a$ is a minimum for the same value of $X$. In terms of $x$, the minimum point is given by $x=\dfrac{-b+\sqrt{b^2-3ac}}{3a}$.

If $k\leqq 0$, that is $b^2\leqq 3ac$, let $k=-m(m\geqq 0)$. Then $Y/a=X(X^2-k)=X(X^2+m)$ is evidently monotonic. Hence, for this case, there is no relative maximum or minimum.

Finally, if $a<0$, the conditions for relative maximum and minimum are reversed; $Y/a$ attaining its maximum when $X=1/3\sqrt{3k}$, and its minimum when $X=-1/3\sqrt{3k}$.

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# A NOTE ON AN *n*th ORDER LINEAR DIFFERENTIAL EQUATION

Murray S. Klamkin

The linear differential equation

$$(1) \qquad (x-a)^n(x-b)^n D^n y = \lambda y, \qquad (a \neq b),$$

has been solved previously by Halphen (see Kamke, Differentialgleichungen, p. 541) by means of the substitution $y = (x-b)^{n-1}F\,(\log\frac{x-a}{x-b})$ which transforms Eq. (1) into a linear one in $F$ with constant coefficients. We effect the solution here in a much simpler way. Also, we find the limit of the solution as $b \to a$.

We assume a solution of the form $y = (x-a)^m(x-b)^{n-m-1}$. On substituting back in Eq. (1) and applying Liebniz's rule for the differentiation of a product, we find that $m$ must be any root $(m_r)$ of the $n$th order polynomial equation

$$(2) \qquad m(m-1)\cdots(m-n+1) = \frac{\lambda}{(a-b)^n}.$$

Whence,

$$(3) \qquad y = \sum_{r=1}^{n} A_r(x-a)^{m_r}(x-b)^{n-1-m_r}.$$

As $b \to a^-$,

$$m_r \approx \frac{\lambda^{1/n}}{a-b}.$$

Consequently,

$$\lim_{b\to a^-}(x-a)^{m_r}(x-b)^{n-1-m_r} = \lim\,(x-b)^{n-1}\left[1-\frac{a-b}{x-b}\right]^{\lambda^{1/n}/(a-b)}$$

$$= (x-a)^n e^{-\omega_r/(x-a)}$$

where $\omega_r^n = \lambda$. For this case Eq. (3) becomes

$$(4) \qquad y = \sum_{r=1}^{n} A_r(x-a)^{n-1}e^{-\omega_r/(x-a)},$$

which satisfies the equation

$$(x-a)^{2n}D^n y = \lambda y. \tag{5}$$

(this equation has also been solved previously, but not in a direct manner by A. Steen, loc. citus. p. 540).

An elegant direct solution to Eq. (5) can be obtained by realizing that the operator $(x-a)^{2n}D^n$ is reducible. This follows from the identity

$$(x-a)^{2n}D^n \equiv \left[(x-a)^2D-(n-1)(x-a)\right]^n. \tag{6}$$

(This identity is easily verified. The reducibility of this operator and generalizations are treated in a paper to be published by D. J. Newman and the author entitled "On the Reducibility of Some Linear Differential Operators".) Using Eq. (6) it follows that

$$\left[(x-a)^2D-(n-1)(x-a)\right] y = \omega y,$$

where $\omega^n = \lambda$. This immediately leads to the solution given by Eq. (4).

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# HARMONIC SETS AND CIRCLES

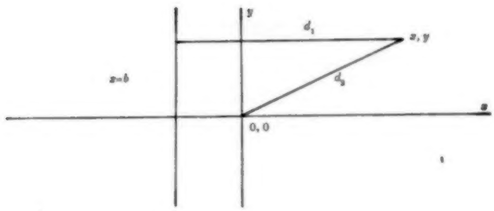
Robert D. Larsson

*FOREWORD*

The purpose of this paper is to reemphasize the fact that even such a simple curve as the circle may be defined as a locus in several ways and that a new definition may lead us to properties, (such as harmonic sets of points), that would not be self-evident under some other definition.

Any circle may be defined as the locus of points such that the distance from a fixed line is equal to the square of the distance from a fixed point, and conversely.

If the fixed line has the equation $x = b$, and the origin is taken as the fixed point then the equation of the circle would be obtained as follows:




$$d_1 = d_2^2$$

$$x - b = x^2 + y^2$$

$$\frac{1}{4} - b = (x - \tfrac{1}{2})^2 + y^2$$

Thus the center of the circle is always ½ unit from the fixed point and the fixed line determines the radius of the circle.

Clearly $b < 1/4$ for a circle with a non-zero radius.

If $b = 0$, $x = b$ is tangent to the circle at the origin.

If $b < 0$, $x = b$ is outside the circle and the origin is inside the circle.

If $0 < b < 1/4$, $x = b$ is outside the circle and the origin is outside the circle.

Obviously to assume any point $x, y$ on the circle as being to the left of the line $x = b$ places the circle in a symmetrical position to the left of the $y$ axis.

Consider the points on the $x$ axis where the line $x = b$ cuts the axis, the origin, the points of intersection of the circle with the $x$ axis and the

center of the circle.

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ |
|---|---|---|---|---|
| $b, 0$ | $0, 0$ | $1/2-\sqrt{1/4-b}, 0$ | $1/2, 0$ | $1/2+\sqrt{1/4-b}, 0$ |

Consider the cross-ratio of any four of these points, and let this cross-ratio be −1.

*Case I*

$$\frac{P_1P_2}{P_3P_2}\times\frac{P_3P_5}{P_1P_5} = -1$$

$$\frac{0-b}{0-\frac{1}{2}+\sqrt{\frac{1}{4}-b}}\times\frac{\frac{1}{2}+\sqrt{\frac{1}{4}-b}-\frac{1}{2}+\sqrt{\frac{1}{4}-b}}{\frac{1}{2}+\sqrt{\frac{1}{4}-b}-b} = -1$$

$$\therefore \frac{\sqrt{\frac{1}{4}-b}-\frac{1}{2}+2b}{b} = -1$$

Solving: $b=0$ or $b=2/9$.

The case of $b=0$ gives us only two points. However, $b=2/9$ gives us a harmonic set of points and all circles radius 1/6 have such a harmonic set associated with them.

*Case II*

$$\frac{P_1P_2}{P_4P_2}\times\frac{P_4P_5}{P_1P_5} = -1$$

$$\frac{0-b}{0-\frac{1}{2}}\times\frac{\frac{1}{2}+\sqrt{\frac{1}{4}-b}-\frac{1}{2}}{\frac{1}{2}+\sqrt{\frac{1}{4}-b}-b} = -1$$

$$(2b+1)\sqrt{\frac{1}{4}-b} = b-\frac{1}{2}$$

$$\therefore b=0,\ b=-\frac{1}{2}\pm\sqrt{\frac{1}{2}}$$

$b=0$ gives three points and $b=-\frac{1}{2}+\frac{\sqrt{2}}{2}$ is an extraneous root.

With $b=-\frac{1}{2}-\frac{\sqrt{2}}{2}$ we obtain another harmonic set of points and all circles of radius $\frac{1}{2}\sqrt{3+2\sqrt{2}}$ have such a harmonic set associated with them.

No other cross-ratio leads to a harmonic set of points.

Therefore, the harmonic sets are:

| $P_2$ | $P_1$ | $P_3$ | $P_5$ |
|---|---|---|---|
| $0, 0$ | $2/9, 0$ | $1/3, 0$ | $2/3, 0$ |

The radius of the circle = 1/6 and the center is at 1/2, 0.

| $P_1$ | $P_2$ | $P_4$ | $P_5$ |
|---|---|---|---|
| $-1/2-\sqrt{2}/2, 0$ | $0, 0$ | $1/2, 0$ | $1/2+1/2\sqrt{3+2\sqrt{2}}, 0$ |

The radius of the circle $= 1/2\sqrt{3+2\sqrt{2}}$ and the center is at $1/2, 0$.

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## A RECURRENCE FORMULA SOLUTION TO $dy^2 + 1 = x^2$

Bob Deemer

*Introduction.* In 1657 Fermat offered as a challenge to English mathematicians the Diophantine equation $dy^2 + 1 = x^2$. In 1767 the first general solution was given by Lagrange.[1] Today the student will find the "textbook" solution to be:

"If $x_1, y_1$ is the minimal positive solution of $x^2 - dy^2 = 1$, then a general solution is given by the equation

$$x + y\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

where $n$ can assume any integral value, positive, negative, or zero."[2]

Note that $d$ is not a perfect square if three or more solutions exist, for if $d = (d')^2$ we have

$$x^2 - (d'y)^2 = 1,$$

which has the sole solutions $\pm 1, 0$ since 0 and 1 are the only two squares which differ by 1.

For proofs that $dy^2 + 1 = x^2$ has at least one solution when $y \neq 0$ and that there are an infinite number of solutions if there are two, $x_1, y_1$ and $x_2, y_2$ where $x_1 \neq x_2$ and $y_1 \neq y_2$, I refer the reader to LeVeque's "Topics in Number Theory, Volume I", pages 140-142.

Let us now use the "textbook" solution to find solutions to the equation

$$2y^2 + 1 = x^2.$$

It takes little trial to discover that the minimal positive solution is 3, 2. Then the general solution is given by

$$x + y\sqrt{2} = (3+2\sqrt{2})^n$$

Letting $n = 2$ we get:

$$x + y\sqrt{2} = 9 + 12\sqrt{2} + 8 = 17 + 12\sqrt{2}$$

So 17, 12 is also a solution.

If we were to work out the first one hundred solutions in this manner

our calculations would become uncomfortably involved. I have found a recurrence formula solution which, besides being easier to use, has the advantage of being derived from very elementary considerations. In fact, no training in number theory is needed to comprehend what follows.

*The Recurrence Formula Solution.* Our problem is to find integers $x, y$ that satisfy the equation

$$dy^2 + 1 = x^2 \tag{1}$$

where $d$ is itself an integer but not a perfect square (for reasons previously stated).

Let $x_1, y_1$ be the minimal solution, $y_1 \neq 0$. Then

$$dy_1^2 + 1 = x_1^2, \qquad x_1 > y_1$$

$$d = (x_1^2 - 1)/y_1^2$$

and

$$[(x_1^2-1)/y_1^2]y^2 + 1 = x^2$$

Let $k = x_1/y_1$, $b = 1/y_1$.
Then

$$(k^2 - b^2)y^2 + 1 = x^2.$$

I shall prove that integral solutions are given by the recurrence formulae:

$$x_{n+1} = ky_{n+1} - by_n \tag{2}$$

$$y_{n+2} = (2k/b)y_{n+1} - y_n \tag{3}$$

where $y_1 = 1/b$ and $y_2 = 2k/b^2$.

*PROOF*

I. True for $n = 1$.

If $y_1 = 1/b$, $x_1 = k/b$; if $y_2 = 2k/b^2$, $x_2 = (2k^2 - b^2)/b^2$; and if $y_3 = (4k^2 - b^2)/b^3$, $x_3 = (4k^3 - 3kb^2)/b^3$ then the equation

$$(k^2 - b^2)y^2 + 1 = x^2 \tag{4}$$

is satisfied.

Note also that

$$x_2 = ky_2 - by_1$$

$$y_3 = (2k/b)y_2 - y_1.$$

Thus the system of equations (2), (3), and (4) are true for $n = 1$.

II. If true for $n$, then true for $n+1$.

Assume the equation

$$(k^2-b^2)y_n^2+1 = x_n^2 \tag{5}$$

is true for some value of $n$ and define this proposition as $P(n)$. (Thus $P(3)$ would assert equation (5) is true for $n=3$). *Define* $x_n$ and $y_n$ by their (2) and (3) counterparts:

$$x_n = ky_n - by_{n-1}$$

$$y_n = (2k/b)y_{n-1} - y_{n-2}$$

$P(n)$ asserts that (5) is true for $n$, or, using the $x_n$ equality and collecting terms,

$$b^2y_n^2 - 2kby_ny_{n-1} + b^2y_{n-1}^2 - 1 = 0. \tag{6}$$

$P(n+1)$: By (3), $y_{n-1} = (2k/b)y_n - y_{n+1}$

If this is substituted in (6) and terms are collected,

$$b^2y_{n+1}^2 - 2kby_{n+1}y_n + b^2y_n^2 - 1 = 0.$$

Hence,

$$(k^2-b^2)y_{n+1}^2 + 1 = x_{n+1}^2$$

Thus if (5) is true for $n$, it is true for $n+1$. We have already confirmed the case $n = 1$, thus completing the inductive proof.

It can easily be shown that if (2) and (3) are true, then

$$x_{n+2} = (2k/b)x_{n+1} - x_n. \tag{7}$$

Also, since $k = x_1/y_1$ and $b = 1/y_1$, the following equalities are evident: $(2k/b) = 2x_1$, $(1/b) = y_1$, $(2k/b^2) = 2x_1y_1$, $(k/b) = x_1$, and $(2k^2-b^2)/b^2 = 2x_1^2 - 1$.

If equations (3) and (7) are put in terms of $x_1, y_1$, as are $x_2$ and $y_2$, we may express our general solution as follows:

*Theorem*. If $dy^2+1=x^2$ has the minimal solution $x_1, y_1$, then the general solution is given by the recurrence formulae

$$\begin{aligned} y_{n+2} &= 2x_1y_{n+1} - y_n & x_{n+2} &= 2x_1x_{n+1} - x_n \\ y_1 &= y_1 \neq 0 & x_1 &= x_1 > y_1 \\ y_2 &= 2x_1y_1 & x_2 &= 2x_1^2 - 1 \end{aligned}$$

where $n \geqq 1$.

Let us again use the example $2y^2+1 = x^2$, where $x_1 = 3$ and $y_1 = 2$. We compute several solutions:

$$y_2 = 2x_1y_1 = 12 \qquad x_2 = 2x_1^2 - 1 = 17$$

$$y_3 = 2x_1y_2 - y_1 = 70 \qquad x_3 = 2x_1x_2 - x_1 = 99$$

$$y_4 = 2x_1y_3 - y_2 = 408 \qquad x_4 = 2x_1x_3 - x_2 = 577$$

Each of these may be easily checked.

*Conclusion.* The advantage of the recurrence solution is that it can be given to the student on a lower level. It would also be easier to find $y_{20}$ if one knew $y_{19}$ and $y_{18}$ by using the recurrence formula instead of the "textbook" solution.

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[1]Uspensky and Heaslet, *Elem. Number Theory*, pp. 335.

[2]LeVeque, *Topics in Number Theory, V.I.*, pp. 142.

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Sophomore

# CURRENT PAPERS AND BOOKS

*Edited by H.V. Craig*

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to *H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.*

# A SUPPLEMENTARY NOTE ON SOLUTIONS OF CUBIC EQUATIONS ON A SLIDE RULE

Benjamin L. Schwartz

In Mr. Louis Pennisi's interesting article on slide rule solutions of cubics in the March-April 1958 issue, an omission occurs which may mislead some readers. The actual procedure for solution is passed over quite briefly, and it is not clearly indicated how the steps described may be achieved. It is perhaps worthwhile to explain this operation in greater detail.

Let us consider the case where the cubic has been reduced to the form

$$x(x^2+c) = d, \qquad \text{with } c>0,\ d>0.$$

(This is Case 1 in the paper cited.) Pennisi's procedure is:

"(1) Move the hairline to the number $d$ on the $D$ scale.

(2) Move the slide until the value of $x$ on the $C$ scale is under the hairline and the ... index of the $C$ scale is on the value of $(x^2+c)$ on the $D$ scale."

This is correct, but the reader may wonder how Step (2) is accomplished. Presumably a trial value of $x$ is selected and set under the hairline. An auxiliary calculation (call it Γ) is then performed to get $(x^2+c)$, and the result of Γ is compared with the value on the $D$ scale opposite the index of $C$. If they agree, the problem is finished. But if not (as will almost invariably be the case with the first trial value), there is no indication of what corrective action to take. Furthermore, the auxiliary calculation Γ is not trivial. It involves the squaring of a number read from the slide rule which therefore presumably has three significant digits. Squaring a three-digit number is not a mental operation; apparently either a second slide

rule or a rather inconvenient pencil-and-paper computation is indicated.

These gloomy considerations are in fact spurious. The squaring can be achieved by invoking the $B$ scale on the rule, and the only mental operation required is a subtraction. Also, positive instructions for moving toward the desired root may be formulated.

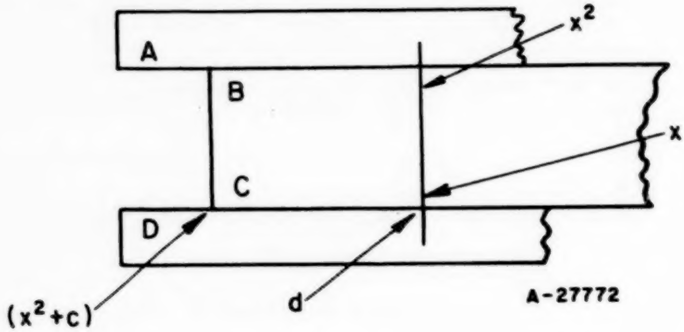


Figure 1.
Slide-Rule Setting for Cubic Equation Solution

Figure 1 shows the desired condition of the slide rule when the solution be achieved. To obtain this condition, the procedure is as follows:

(1) Move the hairline to the number $d$ on the $D$ scale.

(2) Select any trial value of $x$ and move the slide until that value is under the hairline on the $C$ scale.

(3) The trial value of $x^2$ is now under the hairline on the $B$ scale. Denote it by $(B)$. Denote also the number on the $D$ scale opposite the index of $C$ by $(D)$. Mentally compute the trial value of $c = (D)-(B)$.

(4a) If $(D)-(B)$ exceeds the desired value of $c$, the trial root is too small. Move the slide slowly to the left, continually computing $(D)-(B)$ mentally. This trial value of $c$ will decrease monotonically as the slide is moved. Stop the slide when the trial $c$ matches the desired value of $c$.

(4b) If $(D)-(B)$ is less than the desired value of $c$, move the slide to the right and proceed as in (4a).

(5) The desired root now appears under the hairline on the $C$ scale.

The generalization of this procedure to cover the case in which the equation takes the form:

$$x(x^2-c) = d, \qquad \text{with } c>0,\ d>0,$$

is quite apparent. And the steps necessary to process equations with negative values of $d$ are as given in Mr. Pennisi's paper.

It is also worth noting that no catastrophe ensues if the operator forgets

which way to move the slide for Cases (4a) and (4b) above. Should he inadvertently begin to move in the wrong direction, he will immediately notice that his trial value of $c$, $(D)-(B)$, is moving further from the desired value instead of approaching it. He can then reverse the slide direction at once and proceed with the solution.

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Battelle Memorial Institute
Columbus, Ohio

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*Recreational Mathematics*. National Council of Teachers of Mathematics. 1201 Sixteenth Street, N.W., Washington, D.C. $1.20.

This is the second edition of this work. In this edition new topics have been added, antiquated topics deleted, and the total size increased.

*This extensive bibliography should be in every library and on the desk of* every *mathematics teacher. – Ed.*

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*An Introduction to Probability Theory and Its Applications*. By William Feller. John Wiley & Sons, 1957, 461 pages. $10.75.

Volume I of William Feller's "An Introduction to Probability Theory and Its Applications" now appears in a second edition, published in September, 1957, by John Wiley & Sons. The book retains its former general plan and basic content, both of which have achieved such tremendous popularity in only seven years, but adds extensive revisions and stimulating new material that enhance the original substance.

Dr. Feller restricts his subject matter to discrete sample spaces and discrete variables. In this way, he allows himself greater detail in the treatment of many typical problems and explanations of the probabilistic approach to them. In addition to the new chapters, the others include in sequence: the nature of probability theory; the sample space; elements of combinatorial analysis; combination of events; conditional probability and stochastic independence; the binomial and the Poisson distributions; the normal approximation to the binomial distribution; unlimited sequences of Bernoulli trials; random variables and expectation; laws of large numbers; integral valued variables and generating functions; recurrent events and the renewal equation; random walk and ruin problems; Markov chains; algebraic treatment of finite Markov chains; and the simplest time-dependent stochastic processes. Subjects covered in the new chapters are: phenomena of random walks and fluctuation theory; compound distributions

and branching processes.

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Richard Cook

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*Vector Analysis*. By Louis Brand. John Wiley & Sons, 1958, 282 pages. $6.00.

Louis Brand, the distinguished teacher and author of three earlier books, has now written "Vector Analysis" for May publication by John Wiley & Sons.

As Dr. Brand points out, a vectorial treatment of differential geometry, mechanics, hydrodynamics, and electrodynamics is now practically standand procedure. He formulates the theory needed in these fields, and introduces the wide range of its applications.

For the first time in a book on vector analysis, the author covers vector spaces, including Hilbert space, and the volume is one of the few to develop the gradient, divergence, and curl as tensor invariants. The book covers line vectors – both analytical and graphical – and supplies a full discussion of differential invariants and integral theorems, denoting their applications to mathematical physics. Dr. Brand's systematic development of electrodynamics is based on Maxwell's equations; here, he uses the modern mks system of units, fully illustrated with problems. In addition, the book offers a rigorous treatment of Green's theorems and their application to potential theory. Individual chapter headings include: vector algebra, line vectors, vector functions of one variable, differential invariants, integral theorems, dynamics, fluid mechanics, electrodynamics, and vector spaces.

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Richard Cook

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*An Introduction to Combinatorial Analysis*. By John Riordan. John Wiley & Sons, 1958, 244 pages. $8.50.

"An Introduction to Combinatorial Analysis" by John Riordan was published in May by John Wiley & Sons, making available the most up-to-date exposition of this subject. The new book is the latest addition to the Wiley Publications in Statistics (Walter A. Shewhart and S. S. Wilks, Editors).

The author's emphasis is on finding the number of ways there are of

doing some well-defined operation since "anything enumerative is combinatorial" according to Mr. Riordan's definition. The first chapter contains a rapid survey of that part of the theory of permutations and combinations which finds a place in books on elementary algebra. However, Mr. Riordan stresses the relation of these results to generating functions which both illuminates and enlarges them. This leads to the extended treatment of generating functions in Chapter 2, where an important result is the introduction of a set of multivariate polynomials, named after their inventor, E. T. Bell. In Chapter 3, the principle of inclusion and exclusion is covered, while the fourth chapter considers the enumerations of permutations in cyclic representation. Chapter 5 holds a survey of the theory of distribution.

Mr. Riordan's sixth chapter considers together partitions, compositions, and the enumerations of trees and linear graphs. Much of the latter was developed especially for this book and is a continuation of the author's work with R. M. Foster and C. E. Shannon. It is this chapter which also presents Pólya's theorem for the first time in any book, used here to obtain new enumerations of linear graphs and series-parallel networks. The last two chapters of the book are devoted to the enumeration of permutations with restricted position. An extensive problem section, carrying on the text's development, follows each chapter.

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Rose Orente

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*History of Mathematics*. By D. E. Smith. Dover Publications, 1958, 2 Vol. $2.75 per volume, or $5.00 for boxed set.

The word "Algebra" in 16th century England also meant bonesetting. Barbershops often displayed signs reading "Algebra and Bonesetting." This fact comes from D. E. Smith's 2 volume "History of Mathematics" just reprinted by Dover Publications.

Among other interesting facts discussed in the book are: Our division of degrees, hours and minutes into 60 sub-units stems from Babylonia, whose mathematics was based on 60 ... Omar Khayyam is known to the western world chiefly as the author of the Rubaiyat. What is not generally known is that he was one of the most important mathematicians of the 12th century. He seems to have been the first man to work out the binomial theorem ... The Maya Indians, in Central America, worked out place values and the concept of zero hundreds of years before these ideas arose in the Old World ... Newton was so absent minded, or engrossed in mathematical thought, that one day, when he was leading his horse by the bridle,

the horse slipped away, and Newton dragged the empty bridle behind him for hours before someone told him the horse was gone...Mechain, a French mathematician, made an error of 3″ in the latitude of Barcelona. To cover himself he proposed shifting the meridians all over the world ... The Japanese had worked out a calculus independently of European mathematics in the late 17th century ... If it had not been for the Great Fire of 1666 and the necessity of rebuilding St. Paul's, Christopher Wren would be remembered now as a mathematician rather than an architect.

"History of Mathematics" is divided in two major sections. The first volume is chronologically arranged by man and country. The second volume is arranged by subject, discussing the evolution of the different divisions of mathematics. The book is available at your local bookstore or by writing directly to Dover Publications.

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Dan Green

## PROBLEMS AND QUESTIONS

*Edited by*

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should **not** be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

## PROPOSALS

**348.** *Proposed by J.M. Howell, Los Angeles City College.*

If the probability of a team winning a game in a world series is $p$, what is the probability of that team winning the series? What is the probability of winning in 4, 5, 6, or 7 games?

**349.** *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

If $ABCD$, $AEBK$ and $CEFG$ are squares of the same orientations, prove that $B$ bisects $DF$.

**350.** *Proposed by George Bergman, Stuyvesant High School, New York.*

Prove that

$$|x| = x\left(\int_0^x \frac{2e^{1/t}}{(e^{1/t}+1)^2 t^2}\,dt - \frac{2}{e^x+1} + 1\right)$$

for all real non zero $x$.

**351.** *Proposed by D.S. Mitrinovitch, University of Belgrade, Yugoslavia.*

Determine the zeros of the functions $f(z) = \sin az - \sinh bz$ and $g(z) = \cos az - \sinh bz$ where $a$ and $b$ are complex numbers.

**352.** *Proposed by L. Carlitz, Duke University.*

If $P_n(x)$ is the Legendre polynomial, show that:

I. The coefficient of $x^n$ in the polynomial

$$(1-x^2)^n P_n\left(\frac{1+x}{1-x}\right) \quad \text{is equal to} \quad \sum_{r=0}^{n} \binom{n}{r}^3$$

II. The coefficient of $x^n$ in the polynomial

$$(1-x)^{2n}\left[P_n\left(\frac{1+x}{1-x}\right)\right]^2 \quad \text{is equal to} \quad \sum_{r=0}^{n}\binom{n}{r}^4$$

**353.** *Proposed by Karl M. Herstein, New York City, New York.*

Given a line and two points not on the line. Construct two equal circles whose centers are on the given line, which pass through the given points and are tangent to each other.

**354.** *Proposed by Lowell Van Tassel, San Diego Junior College, California.*

A spherical shell is tossed into the air and is shot at by a mathematical archer at infinity. One hemisphere of the shell is painted black. If the archer hits the shell which has been randomly spun, what is the probability that his vector-arrow has either entered or left through a blackened area? Consider the arrow point-sized, the sphere of radius unity and the probability of a hit certain.

## SOLUTIONS

**Errata.** Problem 339, Vol. 31, No. 4, p 229, March - 1958 should read

$$2^{\aleph_0+1}+2^{\aleph_0}=\aleph$$

Solution 326, Vol. 31, No. 5, p 293, May - 1958. The formula for the sum of the generalized harmonic series should read:

$$S=\pi\csc\frac{\pi a}{d}-\frac{2}{d}\sum_{i=1}^{[d/2]}\cos\frac{(2i-1)\pi a}{d}\cdot\ln\sin\frac{(2i-1)\pi}{2d}$$

### Late Solutions

**320.** *Leon Bankoff, Los Angeles, California; Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; M. S. Klamkin, AVCO, Lawrence, Massachusetts; Joe Straley, Chapel Hill High School, Chapel Hill, North Carolina.*

**321.** *Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; M.S. Klamkin, AVCO, Lawrence, Massachusetts.*

**322** and **323** *M. S. Klamkin, AVCO, Lawrence, Massachusetts.*

**324.** *Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; M.S. Klamkin, AVCO, Lawrence, Massachusetts.*

**325.** *D. O. Breault, Station Hospital, Fort Monmouth, New Jersey; Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; M. S. Klamkin, AVCO, Lawrence, Massachusetts.*

**326.** *M. S. Klamkin, AVCO, Lawrence, Massachusetts.*

## A Composite of Conics

**327.** [January 1958] *Proposed by Chih-yi Wang, University of Minnesota.*
Show that the curve

$$x^6 + y^6 - 18(x^4+y^4) + 81(x^2+y^2) - 108 = 0$$

consists of two ellipses and a circle.

**I.** *Solution by M. S. Klamkin, AVCO, Lawrence, Massachusetts.* The equation can be rewritten in the form

$$(x^2+y^2)^3 - 18(x^2+y^2) + 81(x^2+y^2) - 108 + 3x^2y^2(12-x^2-y^2) = 0,$$

or

$$(x^2+y^2-12)\{(x^2+y^2-3)^2 - 3x^2y^2\} = 0$$

or

$$(x^2+y^2-12)(x^2-xy\sqrt{3}+y^2-3)(x^2+xy\sqrt{3}+y^2-3) = 0$$

Whence, the curve consists of 2 ellipses and 1 circle.

**II.** *Solution by A. F. Hordan, University of New England, Armidale, Australia.* Make the equation homogeneous, that is,

$$f = x^6 + y^6 - 18(x^4+y^4)z^2 + 81(x^2+y^2)z^4 - 108z^6 = 0.$$

If this sextic consists of 3 conics, then it has $3\times4 = 12$ double points.
Using the conditions for a double point, namely,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0,$$

we have

$$x^5 - 12x^3z^2 + 27xz^4 = 0$$

$$y^5 - 12y^3z^2 + 27yz^4 = 0$$

$$18z^5 - 9z^3(x^2+y^2) + z(x^4+y^4) = 0.$$

Solve and we obtain the 12 points ($z = 1$ in each case)

$$(0, \pm\sqrt{3})$$
$$(\pm\sqrt{3}, 0)$$
$$(\pm\sqrt{3}, \pm 3)$$
$$(\pm 3, \pm\sqrt{3})$$

The circle passes through the last 8 points (by symmetry). Hence, its radius is $\sqrt{12}$. Thus, the equation of the circle is

$$x^2 + y^2 - 12 = 0.$$

Both ellipses must pass through $(0, \pm\sqrt{3})$, $(\pm\sqrt{3}, 0)$ so their equations reduce to $x^2+y^2+2h'xy-3=0$. One ellipse passes through $(3, \sqrt{3})$, $(\sqrt{3}, 3)$, $(-3, -\sqrt{3})$, $(-\sqrt{3}, -3)$ and the other through $(3, -\sqrt{3})$, $(\sqrt{3}, -3)$, $(-3, \sqrt{3})$, $(-\sqrt{3}, 3)$. We find $2h' = -\sqrt{3}, \sqrt{3}$ respectively. Thus, the equations of the two ellipses are

$$x^2 + y^2 \pm \sqrt{3}\, xy - 3 = 0.$$

Therefore the sextic curve splits up into the two ellipses $x^2 + y^2 \pm \sqrt{3}\, xy - 3 = 0$ and the circle $x^2 + y^2 - 12 = 0$.

*Also solved by Norman Anning, Alhambra, California; George M. Bergman, Stuyvesant High School, New York; Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; J. M. Howell, Los Angeles City College; Joseph D. E. Konhauser, State College, Pennsylvania; F. D. Parker, University of Alaska; Charles F. Pinzka, University of Cincinnati; Lawrence A. Ringenberg, Eastern Illinois University; Sister M. Stephanie, Georgian Court College, New Jersey; P.D. Thomas, Coast and Geodetic Survey, Washington, D.C.; Hazel S. Wilson, Doane College, Nebraska; Dale Woods, Idaho State College and the proposer.*

## Equal Collinear Segments

**328** [January 1958] *Proposed by Walter B. Carver, Cornell University.*

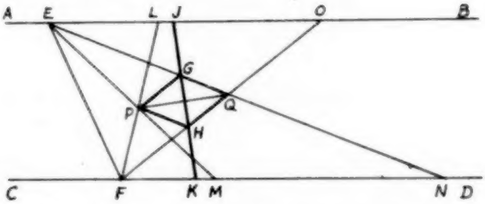
Let $AB$ and $CD$ be parallel lines cut by a transversal at the points $E$ and $F$ respectively. Lines $EM$ and $EN$ trisect angle $FEB$ and $FL$ and $FO$ trisect angle $EFD$ where angle $FEM <$ angle $FEN$ and angle $EFL <$ angle $EFO$. Let $P$ be the intersection of $EM$ and $FL$ while $Q$ is the intersection of $EN$ and $FO$. Through $P$ draw a line parallel to $FQ$ cutting $EQ$ at $G$ and a line parallel to $EQ$ cutting $FQ$ at $H$. The line $GH$ cuts $AB$ at $J$ and $CD$ at $K$. Show that $JG = GH = HK$.

*Solution by Leon Bankoff, Los Angeles, California.* Since $P$ is the incenter of triangle $EFQ$, $PQ$ bisects angle $FQE$. Hence parallelogram $PHQG$

is a rhombus and $GH$ bisects angles $QGP$ and $PHQ$.

Now angle $EFQ$ + angle $QEF = 2\pi/3$. Hence angle $FQE = \pi/3$ and triangles $PHG$ and $HQG$ are equilateral. Then angle $EGJ$ = angle $PGE = \pi/3$ and triangles $EPG$ and $EGJ$ are congruent.

The congruency of triangles $PFH$ and $FKH$ is similarly established, and $HK = PH = GH = PG = JG$.




***Also solved** by J. W. Clawson, Collegeville, Pennsylvania; Huseyin **Demir, Kandilli**, Eregli, Kdz, Turkey; G. I. Gandry, Te Kowai, Via Mackay, **Australia**; Edgar Karst, Endicott, New York; Joseph D. E. Konhauser, **State** College, Pennsylvania and the proposer.*

## A Standard Deviation

**329.** [January 1958] *Proposed by D. A. Breault, Waltham, Massachusetts.* Given $x_i$, normally distributed with fixed mean $\mu$, and some $\epsilon > 0$, determine $\sigma = \sigma(N, \epsilon)$ such that the following holds:

$$P\left[N\left\{\sum_{i=1}^{N} x_i^2\right\} - \left\{\sum_{i=1}^{N} x_i\right\}^2 < \epsilon\right] \geqq .95$$

*Solution by B. K. Gold and J. M. Howell (Jointly) Los Angeles City College.* Let $S^2 = \frac{\Sigma x^2}{N} - \left(\frac{\Sigma x}{N}\right)^2$. Then $N^2S^2 = \Sigma x^2 - (\Sigma x)^2$. Thus

$$P[N^2S^2 < \epsilon] = P\left[\frac{NS^2}{\sigma^2} < \frac{\epsilon}{N\sigma^2}\right].$$

But $\frac{NS^2}{\sigma^2}$ has a Chi-Square distribution with $(n-1)$ degrees of freedom.

This implies that $\frac{\epsilon}{N\sigma^2}$ must be the .05 point of this Chi-Square distribution. For example:

$$N = 2, \frac{\epsilon}{N\sigma^2} > 3.841, \sigma^2 = \frac{\epsilon}{7.681}$$

$$N = 3,\ \frac{\epsilon}{N\sigma^2} > 5.991,\ \sigma^2 = \frac{\epsilon}{17.973}$$

## A Simultaneous System

**330.** [January 1958] *Proposed by M.N.Gopalan, Mysore, India.* Solve

$$x + y + z = a$$

$$x^3 + y^3 + z^3 = b$$

$$x^5 + y^5 + z^5 = c$$

*Solution by Joseph D.E.Konhauser, Haller Raymond and Brown, Inc., State College, Pennsylvania.* We have

$$2a = (x+y) + (y+z) + (z+x),$$

$$a^3 - b = (x+y+z)^3 - (x^3+y^3+z^3) = 3(x+y)(y+z)(z+x), \text{ and}$$

$$a^5 - c = (x+y+z)^5 - (x^5+y^5+z^5) = (5/2)(x+y)(y+z)(z+x)$$

$$[(x+y)^2 + (y+z)^2 + (z+x)^2]$$

$$= \frac{5(a^3-b)}{6}[(x+y)^2 + (y+\ )^2 + (z+x)^2]$$

$$= \frac{5(a^3-b)}{3}[2a^2 - (x+y)(y+z) - (y+z)(x+z) - (x+y)(x+z)].$$

Introduce new variables $u$, $v$, $w$ equal to $x+y$, $y+z$, $z+x$, respectively, then

$$u + v + w = 2a,$$

$$uv + vw + wu = \frac{7a^5-10a^2b+3c}{5(a^3-b)}, \text{ and}$$

$$uvw = \frac{a^3-b}{3}.$$

Denote the right sides by $\alpha$, $\beta$, $\gamma$ respectively, then $u$, $v$, $w$ are roots of the cubic equation $t^3 - \alpha t^2 + \beta t - \gamma = 0$, which is solvable by standard methods. Once $u$, $v$, $w$ are determined it is a simple matter to find $x$, $y$, $z$. The complete set of solutions is obtained by permuting the values of $u$, $v$, $w$.

*Also solved by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey and the proposer.*

## An Integer Problem

**331.** [January 1958] *Proposed by Paul M. Pepper, Ohio State University.*

Show that if $n$ is an integer, then $[2^n + (-1)^{n+1}]/3$ is zero, an odd integer or an odd integer divided by a positive integral power of 2 according as $n$ is zero, positive or negative.

*Solution by Dale Woods, Idaho State College.*
Case I. If $n = 0$ the value is obviously zero.
Case II. $n > 0$

The numerator is not divisible by 2 therefore if it is divisible by three, we must have an odd integer.

If $n = 2m$, then

$$2^{2m} - 1 = 4^m - 1 \equiv 0 \pmod 3$$

therefore the quantity is an integer. If $n = 2m - 1$, then

$$2^{2m-1} + 1 = 2 \cdot 4^m + 1 \equiv 0 \pmod 3$$

therefore the quantity is an integer.
Case III. $n < 0$

Let $n = -k$, $k > 0$ and write the expression

$$\frac{1+(-1)^{1-k}2^k}{3 \cdot 2^k}$$

As in Case II, we have for $k = 2m$,

$$1 - 4^m = 0 \pmod 3$$

and for $k = 2m - 1$,

$$1 + 2 \cdot 4^m \equiv 0 \pmod 3.$$

Hence the value of the expression may be written as an odd integer divided by some power of 2.

*Also solved by Felix A. Beiner, Illinois Institute of Technology; George Bergman, Stuyvesant High School, New York; D.A.Breault, Station Hospital, Fort Monmouth, New Jersey; Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; Joseph D.E.Konhauser, State College, Pennsylvania; James H. Means, Huston-Tillotson College, Texas; William Moser, University of Saskatchewan; F.D.Parker, University of Alaska; Charlès F. Pinzka, University of Cincinnati; C.W.Trigg, Los Angeles City College and the proposer.*

## Polynomial Divisors

**332.** [January 1958] *Proposed by Norman Anning, Alhambra, California.*
Prove that there is no polynomial of degree 22 which is an exact divisor of $x^{45}+1$.

**I** *Solution by L. Carlitz, Duke University.* It is known (see, for example, Mac Duffe's *Abstract Algebra*, p 106) that

$$x^n - 1 = \prod_{d/n} F_d(x),$$

where $F_d(x)$ is a polynomial with integral coefficients of degree $\phi(d)$ and irreducible in the rational field. Thus

$$x^{45} - 1 = F_1 F_3 F_5 F_9 F_{15} F_{45};$$

the irreducible factors on the right are of degree 1, 2, 4, 6, 8, 24. Hence, it is clear that $x^{45}-1$ has no polynomial divisor (with rational coefficients) of degree 22; the same is evidently true of $x^{45}+1$. Note that a divisor of degree 21 does occur.

**II** *Solution by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.* The greatest degree of an exact factor is necessarily the number of imprimitive roots of the given equation of which the roots are all distinct. Since the number $45-\phi(45) = 45-24 = 21$ of the imprimitive roots is less than 22, there will be no such a factor.

*Also solved by D. A. Breault, Station Hospital, Fort Monmouth, New Jersey; C. F. Pinzka, University of Cincinnati; Norman Anning, Alhambra, California and the proposer. One incorrect solution was received.*

## Closing an Expression

**333.** [January 1958] *Proposed by Barney Bissinger, Lebanon Valley College.* Find a closed expression for

$$F(x, y, z) = \frac{x^2}{y}\left\{(z-1) + [2z-(1+2)]\frac{y-x}{y} + [3z-(1+2+3)]\left(\frac{y-x}{y}\right)^2 + \cdots + [nz-(1+2+\cdots+n)]\left(\frac{y-x}{y}\right)n-1\right\}$$

*Solution by Charles F. Pinzka, University of* ***Cincinnati.*** Let $t = \frac{y-x}{y}$ and

$$f(t) = 1 + t + t^2 + \cdots + t^n = \frac{1-t^{n+1}}{1-t}$$

Then

$$F(x, y, z) = \frac{x^2}{y}\left[z f'(t) - 1/2 \left(t f''(t)\right)\right]$$

$$= \frac{x^2}{y}\left[(z-1) f'(t) - \frac{t}{2} f''(t)\right]:$$

Taking the required derivatives

$$f'(t) = \frac{(1-t)(-1)(n+1)t^n + (1-t^{n+1})}{(1-t)^2} = \frac{1-(n+1)t^n + nt^{n+1}}{(1-t)^2} = \frac{y^2}{x^2}[1-(n+1)t^n nt^{n+1}]$$

$$f''(t) = \frac{(1-t)^2[-n(n+1)t^{n-1} + n(n+1)t^n] + 2(1-t)[1-(n+1)t^n + nt^{n+1}]}{(1-t)^4}$$

$$= \frac{-n(n+1)t^{n-1} + n(n+1)t^n - n(n+1)t^{n+1} + 2 - 2(n+1)t^n + 2nt^{n+1}}{(1-t)^3}$$

$$= \frac{2-n(n+1)t^{n-1} + 2(n^2-1)t^n - n(n-1)t^{n+1}}{(1-t)^3}$$

$$= \frac{y^3}{x^3}[2 - n(n+1)t^{n-1} + 2(n^2-1)t^n - n(n-1)t^{n+1}]$$

Now noting that $1 - t = x/y$, we have

$$F(x,y,z) = y(z-1)[1-(n+1)t^n + nt^{n+1}] - \frac{y^2 t}{2x}[2-n(n+1)t^{n-1} + 2(n^2-1)t^n - n(n-1)t^{n+1}].$$

*Also solved by George Bergman, Stuyvesant High School, New York and Joseph D.E. Konhauser State College, Pennsylvania and the proposer.*

## QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

**Q 226.** Maximize $a \cos \alpha + b \cos \beta + c \cos \gamma$ where $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$ [*Submitted by M.S. Klamkin*]

**Q 227.** Find a function $f(n)$ such that $f(1)$, $f(2)$, $f(3)$, $\ldots$, $f(13)$, $f(14)$ be

all prime numbers not all equal nor all distinct. [*Submitted by Huseyin Demir*]

**Q 228.** Show that by reducing the dimensions of a brick one cannot obtain another of half the original volume and half the original surface. [*Submitted by Marlow Sholander*]

**Q 229.** Prove that $x^3+y^3+z^3-3xyz = a^3$ is a surface of revolution [*Submitted by M.S. Klamkin*]

**Q 230.** Each of the letters in $(YE)(ME) = TTT$ uniquely represents a digit. Identify them. [*Submitted by C.W. Trigg*]

## ANSWERS

**A 230.** $TTT$ is a multiple of $111 = 3\cdot 37$. Hence $E = 7$ or $4$. If $E = 4$ then $TTT = (24)(74) > 666$. Therefore $E = 7$ and $(27)(37) = 999$.

**A 229.** The general equation of a aurface of revolution is $(x-a)^2+(y-b)^2+(z-c)^2 = F(rx+sy+tz)$ where the axis is $\frac{x-a}{r} = \frac{y-b}{s} = \frac{z-c}{t}$. Since the given equation may be written as

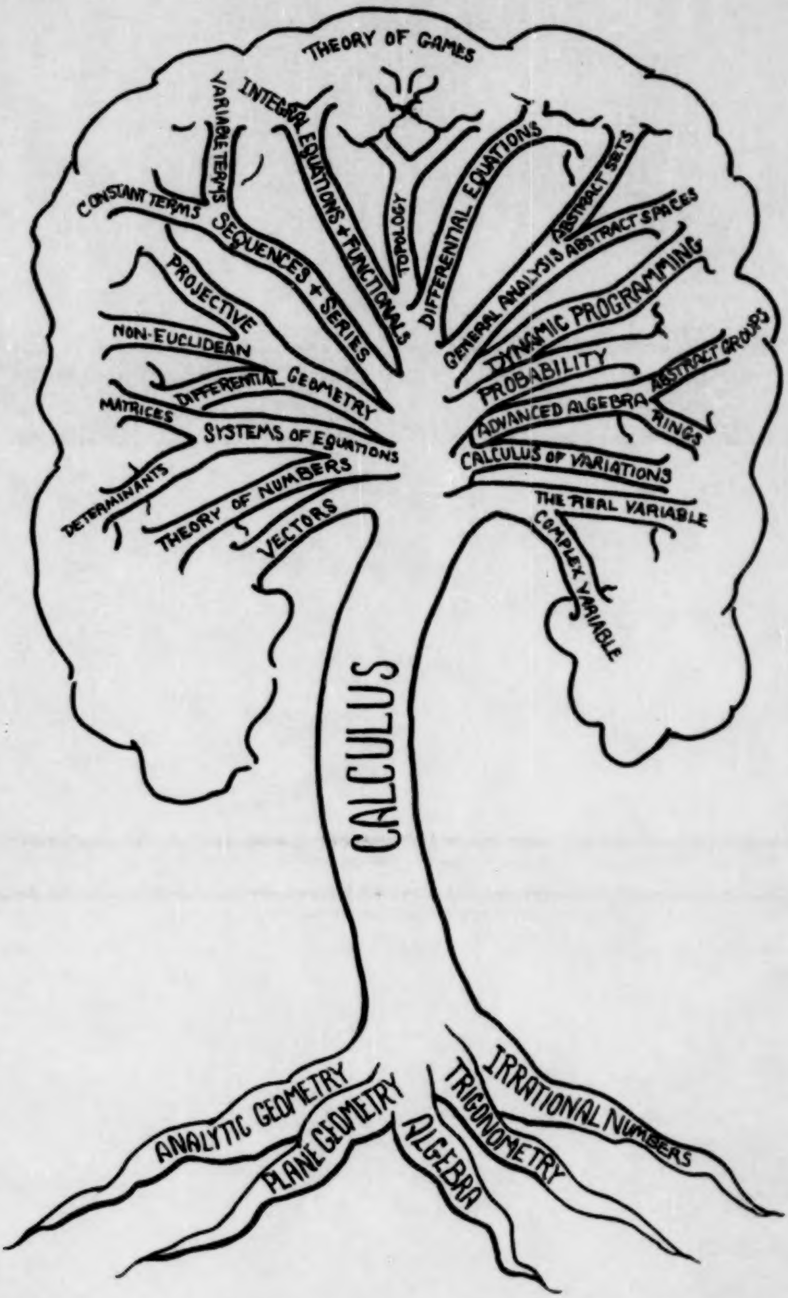
$$x^2+y^2+z^2 = \frac{2a^3}{3(x+y+z)} + \frac{(x+y+z)^2}{3},$$

it follows that it is a surface of revolution about $x = y = z$.

**A 228.** Given $x \leqq a$, $y \leqq b$, $z \leqq c$. On dividing, $ab+bc+ca = 2(xy+yz+zx)$. But $abc = 2xyz$ so we have $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. The contradiction is immediate.

**A 227.** $f(n) = 1-\phi(n)$

**A 226.** If we let $A = ai+bj+ck$ *and* $X = i\cos\alpha + j\cos\beta + k\cos\gamma$ we have to maximize $A\cdot X$ where $|X| = 1$. Consequently, we have max $A\cdot X = |A| = \sqrt{a^2+b^2+c^2}$.



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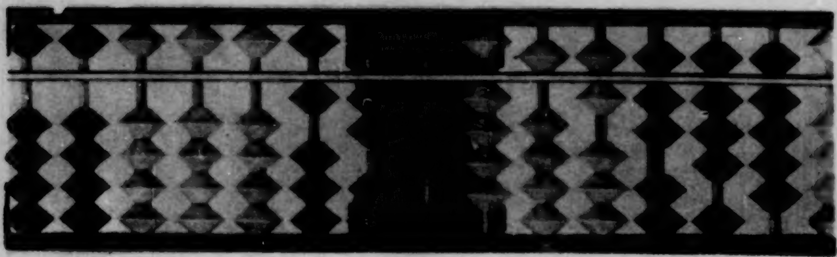

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